PROCEEDINGS

OF THE

Cambridge Philosophical Society.

VOLUME VII.

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OF THE

CAMBRIDGE PHILOSOPHICAL SOCIETY.

VOLUME VII.

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Page 344, line 14, *for* $\frac{Wmg}{772}$ *read* $\frac{W}{772\,mg}$.

PROCEEDINGS

OF THE

Cambridge Philosophical Society.

THE FOUNDATION AND EARLY YEARS OF THE SOCIETY:

AN ADDRESS DELIVERED BY

JOHN WILLIS CLARK, M.A. Trin. Coll. *President*,

ON RESIGNING OFFICE,

27 *October*, 1890.

WHEN the President of a Society lays down his office, it is sual that he should take a more or less extended view of the ast history of the body with which he has been connected, hank his officers, and anticipate a brilliant future from the efforts f his successor. I have no wish to depart from these excellent recedents. I am using no empty phrase when I say that I felt ; a distinguished, and not wholly deserved, honour to be chosen) succeed my dear friend Mr Coutts Trotter—one who by his ıtellectual attainments, by the breadth of his sympathies, and by is unwearied efforts to develop the scientific side of University :aining, was so distinctly marked out as the proper person to reside over a Philosophical Society. I, on the contrary, though he office which I have held for so many years in connection with he New Museums may have enabled me to be a promoter of

science in others, have few claims to be called a man of science myself; and had I not been ably supported by the officers of the Society, the high reputation which we have so long maintained might have been somewhat tarnished by the appearance of a name so humble as mine in the list of Presidents. The Secretaries, however, have taken good care to provide our meetings with papers excellent in matter and varied in character; and the Treasurer has been most energetic and successful in improving our financial position.

The Society has now been in existence for rather more than seventy years; and I may say, without fear of contradiction, that though our position is different from what it was in days which many persons can remember, we still hold our own in public estimation—still exercise a powerful influence in the University. The time has not yet come for the history of the Society to be written; but it has occurred to me that it might be interesting to place on record a few notes respecting its origin, and the early years of its corporate life.

In the Easter Vacation of 1819 Professor Sedgwick—who had been elected to the Woodwardian Chair in the May of the previous year—was taking a tour in the Isle of Wight, and collecting materials for his first course of lectures, which he delivered in the ensuing Easter Term. He was accompanied by Mr Henslow of St John's College, and as the two friends walked and talked they deplored the want of some place in Cambridge to which those interested in science might resort, with the certainty of meeting persons of similar or kindred tastes, and where they might learn what was being done abroad. Their "first idea," we are told, "was to establish a Corresponding Society, for the purpose of introducing subjects of natural history to the Cambridge students"; and on their return to the University they wrote "to their respective friends for their encouragement and support[1]." Easter had fallen late that year (11 April), and therefore the Easter Term would be short. Sedgwick, moreover, was fully occupied with his lectures. The idea too, was novel, and in those days novelty, especially when it took the form of a combination for the prosecution of something foreign to the normal course of study in the place, was sure to encounter disapprobation, if not active opposition. Delay therefore, was unavoidable; and it need excite no surprise that the Michaelmas Term was far advanced before the following notice was circulated, at the suggestion, it is said, of Dr E. D. Clarke Professor of Mineralogy, who entered into the scheme with characteristic enthusiasm, and was always spoken of by Sedgwick as the founder of the Society.

[1] *Memoirs of the Rev. John Stevens Henslow*, by the Rev. L. Jenyns. 8vo. Camb. 1862, p. 17.

CAMBRIDGE, *Oct.* 30, 1819.

The resident Members of the University, who have taken their irst degree, are hereby invited to assemble at the Lecture-Room, under he Public Library, at twelve o'clock, on TUESDAY, Nov. 2, for the urpose of instituting a Society, as a point of concourse, for scientific ommunications.

Hon. Geo. Neville[1].
Bishop of Bristol[2].
Dean of Carlisle[3].
Dr Kaye[4].
Dr Davy[5].
Dr Webb[6].
Dr E. Clarke[7].
Dr Haviland[8].
Dr Ingle[9].
*Prof. Monk[10].
Prof. Cumming[11].
Prof. Sedgwick.
Prof. Lee[12].
R. Woodhouse[13].
Rev. T. Kerrich[14].
Rev. A. Carrighan. [Joh.]
— T. Jephson. [Joh.]
*Rev. J. Holmes. [Pet.]
* — W. Mandell. [Qu.]
* — J. Hustler[15].
* — J. Brown. [Trin.]
— G. Macfarlan. [Trin.]
*W. Hustler[16].
*Rev. J. Lamb[17].
— T. Hughes[18].
* — J. Evans. [Cla.]
* — G. Peacock[19].
— F. Fallows[20].
— J. Whittaker[21].
* — R. Crawley. [Magd.]

[1] Master of Magdalene 1813—53; Dean of Windsor 1845—53.
[2] Will. Lort Mansel, D.D., Master of Trinity 1788—1820; Bp. of Bristol 808—20.
[3] Isaac Milner, D.D., President of Queens' 1788—1820; Dean of Carlisle 1793 -1820.
[4] John Kaye, D.D., Master of Christ's 1814—30; Bp. of Bristol 1820—27; of incoln 1827—53.
[5] Martin Davy, M.D., Master of Gonville and Caius 1803—39.
[6] Will. Webb, D.D., Master of Clare 1815—56.
[7] Edw. Dan. Clarke, LL.D., Professor of Mineralogy 1808—22.
[8] Joh. Haviland, M.D., Regius Professor of Medicine 1817—51.
[9] Tho. Ingle, M.D. Pet.
[10] Ja. Hen. Monk, B.D. Trin., Regius Professor of Greek 1808—23; Bp. of loucester 1830—56.
[11] Ja. Cumming, M.A. Trin., Professor of Chemistry 1815—61.
[12] Sam. Lee, M.A. Queens', Professor of Arabic 1819—31.
[13] Rob. Woodhouse, M.A. Cai., Lucasian Professor 1820—22; Plumian Professor 822—28.
[14] Tho. Kerrich, M.A. Magd., Principal Librarian 1797—1828.
[15] Ja. Devereux Hustler, B.D. Trin.
[16] Will. Hustler, M.A. Jes., Registrary 1816—32.
[17] Master of Corpus Christi 1822—50.
[18] Tho. Smart Hughes, B.D. Emm., Christian Advocate, 1822.
[19] Geo. Peacock, M.A. Trin., Lowndean Professor 1836—58, Dean of Ely 839—58.
[20] Fearon Fallows, M.A. Joh., Director of the Observatory at the Cape of Good ope 1820—31.
[21] Joh. Will. Whittaker, M.A. Joh.

Rev. H. Robinson[22]. J. Henslow[24].
*W. Whewell[23].

It is interesting to remark that the thirty-three persons who signed the above notice differed widely in their pursuits and opinions, and were drawn from eleven Colleges. Among them are six Heads, six Professors, (of Mineralogy, Geology, Chemistry, Medicine, Greek and Arabic), and eleven tutors, or assistant-tutors[25]. It is clear, therefore, that from the first there was nothing sectarian about the Society; it represented no clique; its supporters were not distinguished by any singularity of dress, demeanour, or speech; they merely recognised the need of extending the studies of the University in a scientific direction.

No detailed report of the proceedings at this preliminary meeting was drawn up, but on the next day a brief memorandum was circulated in the University. It ran as follows:

CAMBRIDGE, *Nov.* 3, 1819.

At a Meeting of the Members of this University, which took place on Tuesday, November 2, in the Lecture-Room under the Public Library, in consequence of a requisition to that effect, signed by a number of distinguished Individuals of the different Colleges, the following Resolutions were carried unanimously:

1.—That Dr Haviland be called to the chair.

Proposed by Dr Clarke, and seconded by Mr Kerrich.

2.—That a Society be instituted as a point of concourse for scientific communication.

Proposed by Prof. Sedgwick, and seconded by Mr Robinson.

3.—That a Committee be appointed, consisting of the following gentlemen, who shall report to all Members of the University desirous of belonging to the said Society, such regulations as shall appear to them to be proper for the proposed institution:

Dr Kaye.	Prof. Sedgwick.
Dr Clarke.	Mr Bridge.
Dr Haviland.	Mr Jephson.
Prof. Farish.	Mr Fallows.
Prof. Cumming.	

Proposed by Prof. Monk, and seconded by Mr Hughes.

[22] Hastings Robinson, M.A. Joh.

[23] Professor of Mineralogy 1828—32; of Moral Philosophy 1838—55; Master of Trinity 1841—66.

[24] Joh. Stevens Henslow, M.A. Joh., Professor of Mineralogy 1822—28; of Botany 1825—61. Mr Henslow did not formally resign the Professorship of Mineralogy on obtaining that of Botany until the mode of election had been settled by Sir J. Richardson's award. *Memoirs of Henslow*, ut supra, p. 29.

[25] To the names of these an asterisk has been prefixed in the above list.

4.—That the thanks of the Meeting be given to Dr Haviland, for his able conduct in the chair.

Proposed by Prof. Farish, and seconded by Mr Hughes.

N.B. It is requested that all those gentlemen who are desirous of adding their names to the Society previously to the next Meeting, will signify their intention to the Members of the Committee.

The Committee to whom this important duty was entrusted must have set about their work without delay, for in less than a week the following "Regulations" had been drawn up. The paper containing them is endorsed: "Report of the Committee appointed to form the regulations of a Society to be instituted in this University for Philosophical Communication; to be read at the first meeting of the Society, on Monday, November 15, at one o'clock, in the Lecture Room under the Public Library."

CAMBRIDGE, *November* 8, 1819.

At a Meeting of the Committee appointed to form the regulations of a Society, to be instituted in this University, for Philosophical Communication, it was resolved:

1. That the Society bear the name of The Cambridge Philosophical Society.

2. That this Society be instituted for the purpose of promoting Scientific Enquiries, and of facilitating the communication of facts connected with the advancement of Philosophy.

3. That this Society consist of a Patron, a President, a Vice-President, a Treasurer, two Secretaries, Ordinary and Honorary Members.

4. That a Council be appointed, consisting of the above-mentioned Officers, and five Ordinary Members; three of whom constitute a Quorum: and that no person under the standing of M.A. be of the Council.

5. That the Officers of the Society, with the exception of the Patron, be annually elected by Ballot.

6. That Ordinary Members be chosen from the Graduates of this University by ballot; their Election being determined by a majority of two thirds of the Electors present.

7. That any person desirous of becoming a Member, be proposed by three Ordinary Members; and his name hung up in the Society's room, until the third meeting after the proposition has been made.

8. That Honorary Members be proposed by six Ordinary Members, and balloted for accordingly.

9. That the Meetings of the Society be held on a Monday, once in every fortnight during full term. The President to take the chair at seven o'clock, p.m. and to quit it at nine.

10. That the business of each meeting be conducted in the following order:

    1. The minutes of the preceding meeting read and approved.
    2. Notices of new motions presented.
    3. Members proposed.
    4. Members balloted for.
    5. Motions on the minutes brought forward and determined.
    6. Miscellaneous business.
    7. Communications read and presents acknowledged.

11. That all communications be sent to one of the Secretaries.

12. That nothing be published by the Society which has not been previously approved by the Council.

13. That all questions involving a difference of opinion, be determined by a majority of Members at the next meeting.

14. That if the numbers of the votes be equal, the Chairman have a casting vote.

15. That the annual election of the Officers take place, and the accounts of the Treasurer be passed, at the last meeting of the Society for the year, in the Easter Term.

16. That a Special General Meeting may at any time be called by the Secretary, in consequence either of instructions received from the President, or of a requisition signed by three Ordinary Members. The object must be stated to the Secretary, who shall give to each Member an intimation thereof, stating, three days previously, the time and place of meeting.

17. That the annual Subscription for each member be One Guinea, to be paid in advance, or in lieu of it, a payment of Ten Guineas.

18. That all persons becoming Members, after the first meeting in 1820, pay an admission fee of Two Guineas.

19. That Members be at liberty to introduce each a Visitor; besides whom, the President, for the time being, may admit any person, with the limitation specified in the succeeding Resolution.

20. That no resident Member of the University be allowed to attend more than two meetings of the Society without becoming a Member.

At the meeting which took place on the 15th November, these draft Regulations were adopted, with some slight changes and additions, which are not without interest:

In Rule 4 the number of ordinary members of the Council was increased to seven. To Rule 5, after the words "*by ballot*," was added: "but that the President and Vice-President shall not be eligible for more than two years successively; and that the three Senior Ordinary Members of the Council be changed every year." To Rule 7 was added: "But that all Noblemen, Heads of Houses, Doctors, and Professors, be ballotted for when they are proposed." Rule 18 was omitted; and a new Rule 20 was drawn up: "That the Chancellor of the University be requested to accept the office of Patron."

The Regulations, as altered, having been adopted by the meeting, the Council of the Society for the ensuing year was elected:

PRESIDENT.

The Rev. W. Farish, B.D. Magd. Coll., Jacksonian Professor.

VICE-PRESIDENT.

John Haviland, M.D. St John's Coll., Regius Professor of Physic.

SECRETARIES.

The Rev. A. Sedgwick, M.A. Trin. Coll., Woodwardian Professor.
The Rev. S. Lee, M.A. Queens' Coll., Professor of Arabic.

TREASURER.

The Rev. B. Bridge, B.D., Fellow of Pet. Coll.

ORDINARY MEMBERS OF THE COUNCIL.

The Rev. E. D. Clarke, LL.D. Jes. Coll., Professor of Mineralogy.
—— J. Catton, B.D. Fellow of St John's College.
—— T. Turton, B.D. Fellow of Catharine Hall.
—— T. Kerrich, M.A. Magd. Coll., Principal Librarian.
R. Woodhouse, M.A. Fellow of Caius College.
The Rev. J. Cumming, M.A. Trin. Coll., Professor of Chemistry.
R. Gwatkin, M.A. Fellow of St John's College.

We have seen that the meeting at which the above Regulations were adopted is called "the first meeting of the Society." The Minute Book of the Society, however, takes a different view, and places the birthday of the Society a few weeks later. We there read:

"Minutes of the first meeting of the Cambridge Philosophical Society held in the Museum of the Botanic Garden, Monday, December 13, 1819, Professor Farish in the chair."

At this meeting Professor Farish delivered an address, as also did Dr E. D. Clarke. His biographer says:

"Of this scheme [of founding a Philosophical Society at Cambridge] whose direct object was the promotion of science, and its natural tendency to raise the credit of the University, Dr Clarke was of course one of the earliest and one of the most zealous promoters; and as it was thought advisable, that some address should be provided explanatory of the design and objects of the Institution, he was requested by a sort of temporary council to draw it up. Accordingly he undertook the task, and his address having been read at the first meeting, was afterwards printed by order of the Society, and circulated with the first volume of their Transactions; although for some reason it was not connected with the volume. Nor did his anxiety for the support and honour of the Society rest here; he wrote letters to almost all the literary men of his acquaintance, to request their co-operation and support; combated with great spirit in several instances, the opposition that was made to it from others; and during the short remainder of his life, contributed three papers, which were printed in the first volume of their Transactions[1]."

Dr Clarke's address is brief, and is chiefly occupied with pointing out the advantage of having a society to gather together scientific observations which, if scattered through journals, might escape notice altogether. It concludes with the following practical suggestions:

"Having thus set before the Society the main design and objects of its Institution, the Council beg to call the attention of this Meeting to considerations of a subordinate nature. It will be necessary to provide some place in which the future Meetings may be held, and where a repository may be found for the preservation not only of the archives and records of the Society, but also of such documents, books, and specimens of *Natural History*, as may hereafter be presented or purchased. The utmost economy will at present be requisite in the management of the Society's funds; and therefore if the consent of the University could be obtained it would be highly desirable that the expenses of printing the Society's Transactions should be defrayed by the University. His Royal Highness the Chancellor[2] has accepted of the Office of Patron, and his Letter, containing the expression of his approbation, will be read by one of the Secretaries. The present Vice-Chancellor[3]; our High Steward[4]; both our Representatives in Parliament[5]; and many other distinguished Members of the University, who are not resident, have also contributed towards the undertaking; and there is therefore every reason to hope that the Graduates of this University, who associated for the Institution of the CAMBRIDGE

[1] *The Life and Remains of the Rev. E. D. Clarke, LL.D.* By W. Otter. 4to. Lond. 1824, p. 650.

[2] H. R. H. the Duke of Gloucester.

[3] Mr Serjeant Frere, Master of Downing College.

[4] Lord Hardwicke.

[5] Viscount Palmerston, M.A. St. John's Coll., and J. H. Smyth, M.A. Trin. Coll.

PHILOSOPHICAL SOCIETY, by their assiduity and diligence in its support, and by their conspicuous zeal for the honour and well-being of the University, will prove to other times that their *Lives*, and their *Studies*, have not been in vain."

At this meeting the designation of the Society was altered. The third Minute runs:

"That the words 'and Natural History' be added to the second regulation, which will then stand as follows, viz. 'That this Society be instituted for the purpose of promoting scientific enquiries and of facilitating the communication of facts connected with the advancement of Philosophy and Natural History.'"

The change is slight, but not unimportant, for it determined, for many years, the direction of the Society's labours. Before long, thanks to the enthusiasm and industry of Professor Henslow and Mr Leonard Jenyns, it commenced the formation of a Museum, long the only Zoological Museum in Cambridge; and the legitimate parent of those collections which I may venture to describe as among the most valued possessions of the University.

The Society was now fairly launched; the Syndics of the University Press undertook to publish the *Transactions* free of charge; the number of members increased so rapidly that before the end of 1820 it had reached 171; the finances were in so flourishing a condition that £300 was invested in the funds[1]; and opposition gradually died away. "Among the senior members of the University," wrote Sedgwick to Herschel, 26 February, 1820, "some laugh at us; others shrug up their shoulders and think our whole proceedings subversive of good discipline; a much larger number look on us, as they do on every other external object, with philosophic indifference; and a small number are among our warm friends[2]."

It was further agreed at the first meeting of the Society: "that the High Steward of the University, and the Vice-Chancellor for the time being be Vice-Patrons of the Society"; and at the second meeting: "that the members of the Cambridge Philosophical Society be designated by the name Fellows of the Cambridge Philosophical Society." Early in the following year Dr Clarke proposed: "that the Society be hereafter styled The Cambridge Philosophical and Literary Society." This proposal was not adopted, as I have always thought, most unfortunately. The name would have cemented a connexion between science and literature from which both would have reaped considerable advantage. As time went on the *Transactions* of the Society would probably have had a literary division, as is not uncommon on the continent; and the first object

[1] *Minutes of the Society*, 21 February, 1820.
[2] *Life of Rev. A. Sedgwick*, Vol. I. p. 209.

of the Society's formation—the gathering together of observations and researches that would otherwise be scattered and lost, would have been promoted.

For a few months the Society met in the lecture-room of the building on the east side of what was then the Botanic Garden, built in 1784 for the use of the Professor of Botany and the Jacksonian Professor, and now used by the Professor of Pathology. The selection of Professor Farish as the first President doubtless determined this place of meeting. It was, however, obvious, as Dr Clarke had pointed out, that the Society must have a home of its own as soon as possible. In April, 1820[1], arrangements were made for securing the use of a house in Sidney Street, opposite to Jesus Lane. The Society entered into occupation without delay, and at once commenced the formation of a Museum and a Library; for among the *Minutes* of the first meeting "held in the new rooms," 1 May, 1820, we find:

"The thanks of the Society voted to Mr Henslow for his liberal donation of a valuable collection in some departments of Natural History; and cabinets ordered to be procured for the reception of the specimens."

And at the next meeting (15 May):

"The thanks of the Society voted to Dr Clarke, Dr Haviland, and Mr Bridge for books presented by them to the Society."

Again, 13 November, 1820:

"The thanks of the Society voted to Mr Henslow for a valuable collection of British Insects and Shells systematically arranged in the new cabinet."

The enthusiasm of those days of youth and hope is amusingly illustrated by a notice of motion handed in by Dr Clarke: "that communications announcing discoveries take the precedence of all others." This was agreed to, in a slightly different form, 13 November, 1820.

The Society was barely two years old when a project was started for giving it a social as well as a scientific side, by establishing a reading-room, amply stocked with newspapers, reviews, and magazines, both English and foreign, as well as with scientific journals. A meeting to carry out this scheme was held 22 May, 1821; and so warmly was it taken up that before the end of the year it was agreed that: "the establishment and funds of the Reading-room shall be considered as under the control of the Society." A committee consisting of the Treasurer (Mr Bridge),

[1] *Minutes of the Society*, 17 April, 1820.

Mr Carrighan (Joh.), Mr Griffith (Emm.), Mr Peacock (Trin.), Mr Crawley (Magd.), Mr Whewell (Trin.), Mr Henslow (Joh.), was appointed to draw up the following regulations for the management of it, which the Society adopted, 25 March, 1822.

Rules and Regulations of the Reading Room.

1. Any Fellow of the Society elected before the 1st of January, 1822, may become a member of the Reading-room by writing his name in the book for that purpose and paying the subscription of the current year.

2. Every Fellow of the Society elected after the 1st January, 1822, is a member of the Reading-room during residence.

3. Every Fellow of the Society after becoming a member of the Reading-room continues so during residence.

4. Every member of the Reading-room shall pay an annual subscription of one guinea to the Society, the subscription to be due on the 1st January for the current year, or he may become a member for life by paying ten guineas for the use of the Reading-room in any one year.

5. The following publications shall be taken in[1].

6. Every alteration proposed in the list of publications taken in, shall be signed by at least three members of the Reading-room, read at a meeting of the Society, and suspended in the Room for a fortnight during full Term, for any member to signify his assent or dissent. If the majority in its favour amount to one-third of the signatures and the Council determine that the funds will permit, the alteration shall take place.

7. No Newspaper shall be taken out of the room, and no periodical publication shall be removed, before a succeeding number has appeared.

8. Any member, upon taking out a book, shall give to the servant of the house, a paper, with the title of the book, signed and dated by himself.

9. Any member violating this rule shall pay a fine of 10*s*.

10. Every publication taken out to be returned in a fortnight under a penalty of 2*s*. 6*d*.

11. Any member having lost or damaged a book or paper shall replace it by a fresh copy of the same.

[1] A space of nearly a full page is left in the Minute Book for the list of publications, but it has never been written in.

12. The Reading-room shall be open every day from 8 o'clock in the morning to 10 at night.

13. Strangers may be introduced by a member, but no person resident in Cambridge can be introduced to the room.

14. Non-resident Fellows of the Society when visiting Cambridge shall be entitled to the use of the Reading-room.

15. A Steward[1] shall be appointed at the Annual Meeting of the Society and considered as a member of the Council.

16. The office of the Steward shall be: to procure and take care of the books, to see that the papers are filed, and the room properly prepared for the reception of the members: to collect the bills and to sign them before they are paid by the Treasurer.

It should be remembered that in those days Combination Rooms were ill-supplied with newspapers, and the few that were taken in were generally in the hands of the Senior Fellows. Moreover, in some colleges at least, the juniors were not allowed to use the Combination Room at all, except on Feast Days. The opportunity therefore, of having access at all times to a well-stocked reading-room was eagerly embraced, and formed, with many persons, one of the principal inducements to join the Philosophical Society.

At the beginning of 1832 it became known that the Society would be deprived of the occupation of their house at Midsummer, 1833; nor could another, equally suitable, be either hired or purchased. Under these circumstances it was decided (7 April, 1832), mainly through the influence of Mr Peacock, to apply to St John's College for the lease of a site at the corner of All Saints' Passage, on which the Society might erect "a house of their own, built expressly to suit the objects of the institution." As a preliminary to what the Minutes rightly call "this considerable undertaking," it was decided to obtain a Charter of Incorporation. The Fellows of the Society were evidently warmly in favour of these proposals. A sum of three hundred pounds was subscribed in less than a month to defray the cost of the Charter; and at a special general meeting held 5 May, 1832, the Council was directed (1) to prepare a petition for a charter; (2) to apply to St John's College for a building-lease; (3) "to apply to Mr Humfrey[2] for working-plans, and complete estimates for the New House for the Society, the

[1] The Rev. W. Whewell was Steward of the Reading-room from 1822 to 1826, when he was succeeded by the Rev. Joh. Lodge, University Librarian. He held the office till 1832, when it was discontinued, and a third Secretary was appointed, with the understanding that he should have charge of the Reading-room.

[2] A local builder, who obtained the confidence of the University at this period. He erected the buildings for Human Anatomy. *Architectural History*, Willis and Clark, Vol. III. p. 156.

same to be submitted hereafter to a general meeting of the Society for its approval." Lastly, it was resolved: "that the money requisite for building the Society's house be raised among the members of the Society by shares of £50 each, bearing interest at the rate of four per cent. per annum." So eager was the Society to begin, that it was decided not to wait for the Charter; the plans were approved somewhat hastily, and at a special meeting held 16 May, 1832, the architect was directed to invite tenders.

Early in the Michaelmas Term of 1832 the Charter arrived. Professor Sedgwick happened to be President, and, in order to avoid additional expense in fees[1], it had been agreed that his name alone should appear upon the document. It therefore begins: "Whereas Adam Sedgwick, Clerk, Master of Arts [etc], has by his petition humbly represented unto Us, That he, together with others of our loyal subjects, Graduates of the said University, did in the year One thousand, eight hundred, and nineteen, form themselves into a Society," and so forth. No man had a better right to occupy so prominent a position; and it will be readily understood what pleasure he himself derived from seeing it there. He was never tired of telling the story of the Charter, when, as he put it, "I was the Society."

A special meeting was summoned, 6 November, 1832, to accept the Charter. Sedgwick read it, together with an abstract of it—and it is almost needless to record that it was accepted unanimously. The Council was then directed to prepare a body of Bye Laws—the code by which, with only a few slight alterations, we are still governed.

It was on the occasion of the reception of the Charter that the first of those dinners was held which have now become an annual institution. It seems to me that Sedgwick and the Council of that year wished that November 6, 1832, should be kept as the birthday of the Society—to commemorate the fact that on that day it had assumed a corporate existence. I need not remind you that such a decision involves a sacrifice of twelve years of the Society's life; but, on the other hand, it commemorates an important event in its history—for I believe I am right in saying that it was the first Society out of London to which a Royal Charter was conceded.

The new house was ready for the occupation of the Society in the autumn of 1833. The situation was convenient, and it was itself spacious and well-arranged, with a large meeting-room, museum, and reading-room. The change inaugurated an era of prosperity which lasted for several years. The meetings were well-attended—indeed the Monday evenings on which the Society met were held, by common consent, to be pre-occupied, and no rival attractions were allowed to interfere with them;—The

[1] The fees amounted to £271. *Minutes*, 6 November, 1832.

museum grew apace, under the fostering care of Professor Henslow and his friends; and the reading-room became more popular than ever—a sort of club in fact—where many members of the University passed several hours of each day, reading and writing or conversing with their friends.

I will next quote an excellent account of the Society's Museum, contributed by Mr Leonard Jenyns, in 1838, to *The Cambridge Portfolio.*

The Cambridge Philosophical Society has been employed from the period of its first establishment in 1819, in gradually forming a Museum of Natural History. With a view to this end, it has from time to time effected several purchases, as well as received the contributions of various donors. The Museum however is not large; partly owing to the limited funds which can be appropriated to its support, and partly to the necessarily restricted space allotted for its reception in the Society's house. It is principally, though not exclusively, devoted to the illustration of the British Fauna. The foundation of the Museum may be attributed to Professor Henslow, who presented to the Society at its first institution his entire collection of British Insects and Shells, arranged respectively in two cabinets. Several smaller donations quickly followed, leading the Society to take an increased interest in this part of its establishment. In 1828, a spirited subscription was commenced amongst its members to assist in purchasing a most valuable collection of British Birds, for obtaining which an opportunity then offered itself. This collection had belonged formerly to Mr John Morgan of London. It was extremely rich, especially in the rarer species. Many additions however have been since made to it; and the whole forms now a range of thirty large cases, which are placed round the principal room in the Museum. The birds are beautifully preserved; and the cases of sufficient size to admit, in many instances, of containing entire families. One of the cases contains British Quadrupeds. In 1829, the Society purchased a small collection of British Insects which was incorporated with that previously presented by Professor Henslow. This collection, which consisted of about 2000 species, was valuable from the specimens having been arranged and named by Mr Stephens, the celebrated Entomologist of London. Various additions in the same department have been since made from time to time by different contributors. In 1833, the Society purchased Mr Stephen's entire collection of British Shells, contained in two cabinets and comprising a most extensive series of species as well as of individuals of each. The Museum has been further enriched, in the department of the British Fauna, by a collection of Birds' Eggs, presented in part by Mr Yarrell and in part by Mr Leadbeater;—also by a collection of Fish, obtained principally on the southern shores of the island by Professor Henslow and the Rev. L. Jenyns;—and by a small collection of marine *Invertebrata*, obtained at Weymouth by the former of the two gentlemen last mentioned.

The foreign department of the Museum is not extensive, consisting for the most part of single specimens which have been presented at different times by different individuals. It contains, however, a small collection of reptiles presented by Mr Thomas Bell. It is also rich in Ichthyological specimens; having been presented some years back with a collection of fish made at Madeira by the Rev. R. T. Lowe; subsequently, with another collection made in China by the Rev. G. Vachell; and yet more recently, with the entire collection of Fish brought home from South America and some other portions of the globe by C. Darwin, Esq., of Christ's College, and accompanying Naturalist in the late voyage of the Beagle, under the command of Captain Fitzroy. The whole of the fish above alluded to, as well as those belonging to the British collection, are preserved in spirits. They amount to several hundred species; and many of those comprised in the Darwin collection are entirely new. Altogether, they constitute a highly valuable as well as interesting portion of the Society's Museum.

Independently of the collections above enumerated, the Philosophical Society has made it an object to establish a separate collection of the principal animals found in Cambridgeshire. This is a step of the utility of which there can be no doubt. Local collections of this nature tend to illustrate the Faunas of particular districts; and local Faunas offer the best materials for completing our knowledge of the Zoology of the whole kingdom. They also throw light upon the geographical distribution of animals. In proportion to the number of places in which such collections are established, they assist in determining the extreme range of the different species, as well as the districts to which they are ordinarily confined. In this department, however, the Birds of Cambridgeshire and a few of its Mammalia are alone as yet fitted up for public inspection; but considerable collections have been made in the other classes, which are destined one day to take their place in the Museum also.

The Museum of the Society, and that part of it in particular which has been just alluded to, has been probably instrumental in exciting much interest in the University in the science of Zoology, and diffusing amongst its members a taste for such pursuits. Nor is the surrounding neighbourhood at all unfavourable for the researches of the naturalist. On the contrary, Cambridgeshire may be considered as rich in animal productions. From combining within itself a considerable variety of soil and situation, it adapts itself to the habits of very different species. The fens in particular are inhabited by many rare aquatic birds and insects; and some of these, previous to the introduction of the present system of drainage, were in considerable abundance. It may perhaps be interesting to mention, that the entire number of vertebrate animals found in Cambridgeshire amount to 281. Of these 38 belong to the class Mammalia; 204 to that of Birds; 9 to that of Reptiles; and 32 to that of Fish. The invertebrate animals require further investigation; but they probably exceed 9,000, of which the greater portion belong to the division of Annulosa.

The Society has a small collection of minerals and fossils; but there

being other Museums in the University devoted to these departments, they have received less of its attention than the Zoological part of the Museum above noticed. There are also a few antiquities, some of which were obtained in the county.

The Society's house had been built, to a great extent, with borrowed money, as I have related, and it had cost a far larger sum than had been anticipated. It was possible to pay the interest on the loans, but the Society found itself unable to establish a sinking-fund for the repayment of the capital. Moreover, the number of Fellows gradually decreased. At one time it was usual for nearly every Fellow of a College to become a Fellow of the Philosophical Society; but, when the novelty of the existence of such a body in Cambridge had worn off, and when the reading-room had several rivals, not to mention the reduction of the price of newspapers, which enabled them to be taken in at home—there seemed to be no special reason for joining a Society where the papers read were chiefly mathematical, and which offered no other attractions not to be found elsewhere. The officers of the Society did their best in these adverse days; and some of those who had lent money cancelled their bonds—as for instance Professor Peacock, Professor Sedgwick, Professor Adams, and Professor Babington; but the financial difficulty could not be overcome. Finally, in 1865, the Museum was offered to, and accepted by, the University[1]; the house was sold; and the Society found a home at the New Museums[2].

In this brief review I have of necessity omitted much that I should have been glad to record, had I not determined to write a sketch and not a history. I cannot, however, conclude without drawing attention to our publications. No one, I think, can look through the volumes of *Transactions* and *Proceedings* without admitting that the papers therein printed or abstracted will hold their own in originality and value against those of almost any society. The *Proceedings*, as you are aware, do not begin before 1843. I have therefore appended to this paper brief notices of the communications made before that date, as recorded in the Minute Book. These will, I feel sure, be found interesting. They show what some of the best men in the place were working at; and they testify to the genuine interest taken by them in the Society. Whatever they did, they hastened to communicate it, though, to our great loss, they too often neglected to prepare their work for our *Transactions*. I have also prepared a list of the Presidents, Secretaries, and Treasurers, from the beginning to the present time.

[1] Grace, 24 May, 1865. [2] Grace, 8 June, 1865.

OFFICERS OF THE SOCIETY.

PRESIDENTS.

Date of Election.	
15 Nov. 1819.	Rev. Will. Farish, M.A. Magd., *Jacksonian Professor.*
22 May, 1821.	Rev. Ja. Wood, D.D., *Master of S. John's College.*
13 May, 1823.	John Haviland, M.D. Joh., *Regius Professor of Physic.*
17 May, 1825.	Rev. Ja. Cumming, M.A. Trin., *Professor of Chemistry.*
22 May, 1827.	Rev. Joh. Kaye, D.D., *Master of Christ's Coll. and Bp. of Lincoln.*
19 May, 1829.	Rev. Tho. Turton, D.D. Cath., *Regius Professor of Divinity.*
17 May, 1831.	Rev. Adam Sedgwick, M.A. Trin., *Woodwardian Professor.*
6 Nov. 1833.	Rev. Joshua King, M.A. Qu., *President of Queens' College.*
6 Nov. 1835.	Rev. Will. Clark, M.D. Trin., *Professor of Anatomy.*
6 Nov. 1837.	Rev. Joh. Graham, D.D. Chr., *Master of Christ's College.*
6 Nov. 1839.	Rev. Will. Hodgson, D.D. Pet., *Master of Peterhouse.*
6 Nov. 1841.	Rev. Geo. Peacock, D.D. Trin., *Lowndean Professor.*
6 Nov. 1843.	Rev. Will. Whewell, D.D. Trin., *Master of Trinity College.*
6 Nov. 1845.	Rev. Ja. Challis, M.A. Trin., *Plumian Professor.*
6 Nov. 1847.	Rev. Hen. Philpott, D.D. Cath., *Master of S. Catharine's Coll.*
6 Nov. 1849.	Rev. Rob. Willis, M.A. Gonv. and Cai. Coll., *Jacksonian Professor.*
6 Nov. 1851.	Will. Hopkins, M.A. Pet.
7 Nov. 1853.	Rev. Adam Sedgwick, M.A. Trin., *Woodwardian Professor.*
6 Nov. 1855.	Geo. Edw. Paget, M.D. Gonv. and Cai. Coll.
26 Oct. 1857.	Will. Hallows Miller, M.D. Joh., *Professor of Mineralogy.*
31 Oct. 1859.	Geo. Gabriel Stokes, M.A. Pemb., *Lucasian Professor.*
28 Oct. 1861.	Joh. Couch Adams, M.A. Pemb., *Lowndean Professor.*
26 Oct. 1863.	Will. Hepworth Thompson, M.A. Trin., *Regius Professor of Greek.*
30 Oct. 1865.	Rev. Hen. Wilkinson Cookson, D.D. Pet., *Master of Peterhouse.*
28 Oct. 1867.	Rev. Will. Selwyn, D.D. Joh., *Lady Margaret's Professor.*
25 Oct. 1869.	Art. Cayley, M.A. Trin., *Sadlerian Professor.*
30 Oct. 1871.	Geo. Murray Humphry, M.D. Down., *Professor of Anatomy.*
27 Oct. 1873.	Ch. Cardale Babington, M.A. Joh., *Professor of Botany.*
25 Oct. 1875.	Ja. Clerk Maxwell, M.A. Trin., *Professor of Experimental Physics.*

Date of Election.	
29 Oct. 1877.	Geo. Downing Liveing, M.A. Joh., *Professor of Chemistry.*
27 Oct. 1879.	Alf. Newton, M.A. Magd., *Professor of Zoology and Comparative Anatomy.*
31 Oct. 1881.	Fra. Maitland Balfour, M.A. Trin.
30 Oct. 1882.	Ja. Whitbread Lee Glaisher, M.A. Trin.
27 Oct. 1884.	Mich. Foster, M.A. Trin., *Professor of Physiology.*
26 Oct. 1886.	Rev. Coutts Trotter, M.A. Trin.
30 Jan. 1888.	Joh. Willis Clark, M.A. Trin.

SECRETARIES.

15 Nov. 1819.	Rev. Adam Sedgwick, M.A. Trin., *Professor of Geology.*
	Rev. Sam. Lee, M A. Qu., *Professor of Arabic.*
22 May, 1821.	Rev. Geo. Peacock, M.A. Trin.
	Joh. Stevens Henslow, M.A. Joh.
9 May, 1826.	Rev. Joh. Stevens Henslow, M.A. Joh., *Professor of Mineralogy and Botany.*
	Rev. Will. Whewell, M.A. Trin.
6 Nov. 1833.	Rev. Joh. Stevens Henslow, M.A. Joh., *Professor of Botany.*
	Rev. Will. Whewell, M.A. Trin.
	Rev. Joh. Lodge, M.A. Magd., *University Librarian.*
7 Nov. 1836.	Rev. Joh. Stevens Henslow, M.A. Joh., *Professor of Botany.*
	Rev. Will. Whewell, M.A. Trin.
	Rev. R. Willis, M.A. Gonv. and Cai.
6 Nov. 1839.	Rev. Will. Whewell, M.A. Trin.
	Rev. R. Willis, M.A. Gonv. and Cai., *Jacksonian Professor.*
	Will. Hopkins, M.A. Pet.
7 Nov. 1842.	Rev. R. Willis, M.A. Gonv. and Cai., *Jacksonian Professor.*
	Will. Hopkins, M.A. Pet.
	Will. Hallows Miller, M.D. Joh., *Professor of Mineralogy.*
6 Nov. 1851.	Will. Hallows Miller, M.D. Joh., *Professor of Mineralogy.*
	Ch. Cardale Babington, M.A. Joh.
	Geo. Gabriel Stokes, M.A. Pemb., *Lucasian Professor.*
6 Nov. 1854.	Ch. Cardale Babington, M.A. Joh.
	Joh. Couch Adams, M.A. Pemb.
	Rev. Ch. Fre. Mackenzie, M.A. Gonv. and Cai.
6 Nov. 1855.	Ch. Cardale Babington, M.A. Joh.
	Joh. Couch Adams, M.A. Pemb.
	Geo. Downing Liveing, M.A. Joh.
25 Oct. 1858.	Ch. Cardale Babington, M.A. Joh.
	Geo. Downing Liveing, M.A. Joh.
	Norman Macleod Ferrers, M.A. Gonv. and Cai.

Date of Election.	
29 Oct. 1866.	Ch. Cardale Babington, M.A. Joh., *Professor of Botany.*
	Geo. Downing Liveing, M.A. Joh., *Professor of Chemistry.*
	Rev. T. G. Bonney, M.A. Joh.
31 Oct. 1870.	Rev. T. G. Bonney, M.A. Joh.
	Joh. Willis Clark, M.A. Trin.
	Rev. Coutts Trotter, M.A. Trin.
27 Oct. 1873.	Joh. Willis Clark, M.A. Trin.
	Rev. Coutts Trotter, M.A. Trin.
	Rev. Joh. Batteridge Pearson, B.D. Emm.
28 Oct. 1878.	Joh. Willis Clark, M.A. Trin.
	Rev. Coutts Trotter, M.A. Trin.
	Ja. Whitbread Lee Glaisher, M.A. Trin.
30 Nov. 1882.	Joh. Willis Clark, M.A. Trin.
	Rev. Coutts Trotter, M.A. Trin.
	Will. Mitchinson Hicks, M.A. Joh.
29 Oct. 1883.	Rev. Coutts Trotter, M.A. Trin.
	Ri. Tetley Glazebrook, M.A. Trin.
	Sydney Howard Vines, M.A. Chr.
26 Oct. 1886.	Ri. Tetley Glazebrook, M.A. Trin.
	Sydney Howard Vines, M.A. Chr.
	Jos. Larmor, M.A. Joh.
31 Oct. 1887.	Sydney Howard Vines, M.A. Chr.
	Jos. Larmor, M.A. Joh.
	Matth. Moncrieff Pattison-Muir, M.A. Gonv. and Cai.
29 Oct. 1888.	Jos. Larmor, M.A. Joh.
	Matth. Moncrieff Pattison-Muir, M.A. Gonv. and Cai.
	Sidney Fre. Harmer, M.A. King's.
28 Oct. 1889.	Jos. Larmor, M.A. Joh.
	Sidney Fre. Harmer, M.A. King's.
	Andr. Russell Forsyth, M.A. Trin.

Treasurers.

15 Nov. 1819.	Rev. Bewick Bridge, B.D. Pet.
17 May, 1825.	Fre. Thackeray, M.D. Emm.
6 Nov. 1834.	Rev. Geo. Peacock, M.A. Trin.
6 Nov. 1839.	Geo. Edw. Paget, M.D. Gonv. and Cai.
7 Nov. 1853.	Rev. Tho. Hedley, M.A. Trin.
26 Oct. 1857.	Rev. Will. Magan Campion, M.A. Qu.
23 Oct. 1876.	Ja. Whitbread Lee Glaisher, M.A. Trin.
28 Oct. 1878.	Rev. Joh. Batteridge Pearson, D.D. Emm.
29 Oct. 1883.	Joh. Willis Clark, M.A. Trin.
31 Oct. 1887.	Ri. Tetley Glazebrook, M.A. Trin.

COMMUNICATIONS MADE TO THE SOCIETY.

February 20, 1820[1].

By Professor Farish (President): On Isometrical Perspective.
By Prof. E. D. Clarke: On the discovery of Cadmium in some of the English ores of zinc; with some directions respecting the mode of operating.
By Captain Fairfax (presented by Mr Okes): On Soundings at Sea.

March 6, 1820.

By Joh. Hailstone, M.A. (Trin.): On the probable origin of a fossil body found on the coast of Scarborough.
By Professor Farish (President): On Isometrical Perspective (concluded). *Trans.* I. 1—19.
By Joh. Fre. Will. Herschel, M.A. (Joh.): On functional equations. *Trans.* I. 77—87.
By Mr Okes: On some fossil remains of the Beaver, found near Chatteris. *Trans.* I. 175—177.

March 20, 1820.

By Professor Sedgwick: On the Geology of Cornwall, etc.
By Mr Thompson (Joh.): A translation from Gemmellaro's account of the last great eruption of Etna, in 1819. (Presented by Dr E. D. Clarke.)

April 17, 1820.

A letter from the Rev. J. Davis to the Rev. Dr Wood, detailing certain optical phenomena observed at Hilkhampton in Cornwall on Wednesday, April 5th, 1820, was read to the Society.
By Joh. Fre. Will. Herschel, M.A. (Joh.): On the rotation impressed by plates of rock crystal on the planes of polarization of the rays of light as connected with certain peculiarities in its crystallization. *Trans.* I. 43—52.
By Will. Whewell, M.A. (Trin.): On the position of the apsides of orbits of great eccentricity. *Trans.* I. 179—191.

May 1, 1820.

By Professor Farish (President): On the mode of conducting Polar navigation.
By Joh. Fre. Will. Herschel, M.A. (Joh.): On certain remarkable instances of deviation from Newton's scale in the tints developed by crystals with an axis of double refraction on exposure to polarized light. *Trans.* I. 21—41.
By Ch. Babbage, M.A. (Trin): On the Calculus of Functions. *Trans.* I. 63—76.

May 15, 1820.

By Mr Emmett: Researches into the mathematical principles of chemical philosophy.
By Prof. E. D. Clarke: On the chemical constituents of the purple precipitate of Cassius. *Trans.* I. 53—61.
By Professor Sedgwick: On the physical structure of Cornwall, etc. (continued from 23 March). *Trans.* I. 89—146.

[1] The Minute Book says: "Monday, February 21, 1820"; but in 1820 February 21 fell on a Tuesday.

By Sam. Hunter Christie, M.A. (Trin.): On the laws according to which masses of iron influence magnetic needles. *Trans.* I. 147—173.

A letter from the Rev. J. Davis to the Rev. Dr Wood, containing some further details respecting certain optical phenomena mentioned in the Minutes of the Society's Meeting on the 17th of April.

November 13, 1820.

By Prof. E. D. Clarke: On a method of giving to common Paris Plaster casts the appearance of polished Rosso Antico.

By Professor Lee: On certain astronomical tables by Mohammed al Farsi, a MS. copy of which exists in the University Library. *Trans.* I. 249—265.

November 27, 1820.

By Prof. E. D. Clarke: On the discovery of native natron in Devonshire. *Trans.* I. 193—201.

By the same: Notice respecting the sarcophagus brought from Egypt by Mr Belzoni.

By Will. Cecil, M.A. (Magd.): On the application of hydrogen gas to produce a moving force in machinery, with the description of an engine where the moving force is produced on that principle. *Trans.* I. 217—239.

December 11, 1820.

A communication from Dr Wavell (Hon. Member), with some observations by Dr E. D. Clarke on the decomposition of a quartzose rock, and on the formation of natron.

By Professor Haviland, Vice-President: On some unusual appearances presented by the stomach of a man who died of a fever. *Trans.* I. 287—290.

By Professor Lee: A demonstration of the properties of parallel lines, by Nasir el Din, translated from the Arabic.

March 5, 1821.

"Communications from Professor Leslie and Dr Wavell read by Dr E. D. Clarke."

By Fra. Lunn, B.A. (Joh.): On the analysis of a native phosphate of copper. *Trans.* I. 203—207.

By Prof. E. D. Clarke: On the crystallization of water. *Trans.* I. 209—215.

March 19, 1821.

A communication (read by Professor Sedgwick) from Mr Ross respecting some minerals found at Buralston.

By Prof. E. D. Clarke: On Arragonite.

April 2, 1821.

By Joh. Leslie, Professor of Mathematics in the University of Edinburgh (Hon. Member): On the effect of hydrogen gas on the propagation of sound. (Read by Professor Lee.) *Trans.* I. 267.

By Professor Cumming: Exhibition of experiments and communication read on the effects of the galvanic fluid on the magnetic needle. *Trans.* I. 269—279.

By Professor Sedgwick: On the geology of the Lizard district.

May 7, 1821.

By Joh. Fre. Will. Herschel, M.A. (Joh.): On the refraction of Apophyllite. *Trans.* I. 241—247.

By Professor Sedgwick: On the geology of the Lizard (concluded). *Trans.* I. 291—330.

May 21, 1821.

By Professor Cumming: On the connexion between galvanism and magnetism. *Trans.* I. 281—286.

By Will. Cecil, M.A. (Magd.): On the application of regulators to machinery.

November 12, 1821.

The following communication from Dr Brewster was read by Prof. E. D. Clarke:

"I have examined with great care a specimen of Leelite, and I find it to be an irregularly crystallized body, like Hornstone, Flint, and having a sort of quaquaversus structure, or one in which the axes of the elementary particles are in every possible direction, instead of being parallel, as they must be in all regular crystals. The alumina which Leelite contains gives it quite a different action upon light from any of the analogous siliceous substances; and I have thus obtained an unerring optical character by which Leelite may be distinguished from them with the greatest facility.

In examining the different kinds of topazes, I have found that the colourless topazes, and the blue topazes of Aberdeenshire, differ not merely from the yellow Brazil topazes, but also from one another."

Signed, D. Brewster.

By Mr Okes: On a peculiar case of an enlargement of the ureters in a boy of eleven years of age. *Trans.* I. 351—358.

November 26, 1821.

By Fre. Thackeray, M.D. (Emm.): On a remarkable instance of organic remains found on the turnpike road between Streatham and Wilburton in the Isle of Ely. *Trans.* I. 459.

By Will. Mandell, B.D. (Qu.): On an improvement in the common mode of procuring potassium. *Trans.* I. 343—345.

By Will. Whewell, M.A. (Trin.): On the crystallization of fluor spar. *Trans.* I. 331—342.

By Joh. Stevens Henslow, M.A. (Joh.): On the geology of the Isle of Anglesea.

December 10, 1821.

By Professor Cumming: On a remarkable human calculus in the possession of the Society of Trinity College. *Trans.* I. 347—349.

By Joh. Stevens Henslow, M.A. (Joh.): On the geology of the Isle of Anglesea (continued).

By Ch. Babbage, M.A. (Pet.): On the use of signs in mathematical reasoning. (Read by Mr Peacock.) *Trans.* II. 325—377.

February 25, 1822.

By Joh. Hailstone, M.A., late Fellow of Trin. Coll., and Woodwardian Professor: Some observations on the weather, accompanied by an extraordinary depression of the barometer, during the month of December, 1821. (Read by the Secretary.) *Trans.* I. 453—458.

By Joh. Stevens Henslow, M.A. (Joh.): On the geology of the Isle of Anglesea (concluded). *Trans.* I. 359—452.

March 11, 1822.

The President proposed that, in consequence of the death of the Vice-President of the Society, Prof. E. D. Clarke, the meeting should be adjourned without proceeding to the regular business of the evening. This proposition was agreed to unanimously, and the Society adjourned immediately.

March 25, 1822.

By Will. Mandell, B.D. (Qu.): A description of a new self-regulating lamp.

By Mr H. B. Leeson: A description of a safety apparatus to the hydrostatic blowpipe of Tofts, by which it may be converted into an oxyhydrogen blowpipe without danger to the operator. (Read by Mr Peacock.) A model of the safety apparatus, and of the blowpipe, was exhibited to the Society and explained by Mr Leeson.

By Geo. Biddell Airy, student of Trinity College: On the alteration of the focal length of a telescope by a variation of the velocity of light and of the observations to which the change may give rise. (Read by Mr Peacock.)

April 22, 1822.

By David Brewster, LL.D., Honorary Member of this Society: On the difference of optical structure between the Brazilian topazes and those of Scotland and New Holland. *Trans.* II. 1—9.

May 6, 1822.

By Will. Whewell, M.A. (Trin.): On the rotation of bodies. *Trans.* II. 11—20.

By Dav. Brewster (Hon. Member): On the distribution of the colouring matter, and on certain peculiarities in the structure and optical properties of the Brazilian topaz. *Trans.* II. 1—9.

May 21, 1822.

By Professor Sedgwick: On the basaltic dykes in the county of Durham, and the great basaltic formation in Teesdale. *Trans.* II. 21—44.

November 11, 1822.

By Will. Whewell, M.A. (Trin.): On the oscillations of a chain suspended vertically, and on the oscillations of a weight drawn up uniformly by a string.

By Fra. Gybbon Spilsbury: On a peculiar relation existing between gravity and the production of magnetism in galvanic combinations. (Read by Mr Henslow.) *Trans.* II. 77—83.

November 25, 1822.

By Geo. Biddell Airy, Scholar of Trin. Coll.: On the construction of achromatic reflecting telescopes with silvered lenses in the place of metallic mirrors. Read by Mr Peacock. *Trans.* II. 105—118.

December 9, 1822.

By Will. Cecil, M.A. (Magd.): On an apparatus for grinding telescopic mirrors and object-lenses. *Trans.* II. 85—103.

February 17, 1823.

No papers recorded.

March 3, 1823.

No papers recorded.

March 17, 1823.

No papers recorded.

April 14, 1823.

By Will. Whewell, M.A. (Trin.): On the different methods which have been proposed to grind lenses and mirrors by machinery to a parabolic form.

By Joshua King, M.A. (Qu.): A new demonstration of the parallelogram of forces. *Trans.* II. 45—46.

By Geo. Peacock, M.A. (Trin.): On the analytical discoveries of Newton and his contemporaries.

April 28, 1823.

By Professor Cumming: On the development of electro-magnetism by heat. *Trans.* II. 47—76.

May 12, 1823.

By Geo. Peacock, M.A. (Trin.): On the irregular indications of the thermometer.

November 10, 1823.

By Professor Cumming: On rotation produced by electro-magnetism as developed by heat.

A letter was read by Mr Peacock from Will. Joh. Bankes, M.P., on the subject of the manuscript on papyrus of the lost book of the Iliad, recently discovered by one of his agents in the island of Elephantina in Upper Egypt, accompanied by a facsimile made by Mr Salt of the ten first lines of the manuscript.

By Geo. Biddell Airy, B.A. (Trin.): Explanation of an instrument exhibited to the Society, for the purpose of proving by experiment the constancy of the ratio of the sines of incidence and refraction in liquids. (Read by Mr Peacock.)

By Joh. Murray, F.S.A.: Some remarks on the temperature of the egg, as connected with its physiology. (Read by Mr Peacock.)

By the same: Experiments and observations on the temperature developed in voltaic action, and its unequal distribution. (Read by Mr Peacock.)

November 24, 1823.

By Will. Whewell, M.A. (Trin.): On the expressions for the cosine of the angle between two lines and two planes when referred to oblique coordinates. *Trans.* II. 197—202.

By Olinthus Gregory, LL.D.: An account of some experiments made in order to ascertain the velocity with which sound is transmitted in the atmosphere. (Read by Mr Peacock.) *Trans.* II. 119—137.

December 8, 1823.

By Professor Cumming: Exhibition of Dobereiser's experiments of the contortion of platina wire by a stream of hydrogen gas.

By Will. Cecil, M.A. (Magd.): Exhibition of a model of an improved ear-trumpet.

By Geo. Peacock, M.A. (Trin.): On the analytical discoveries of Sir I. Newton. (Concluded.) Mr Peacock read three unedited letters of Newton to Dr Keill on the subject of the controversy on the discovery of the method of fluxions.

March 1, 1824.

By Mr Okes: Notice of a magnificent collection of fossil bones, found near Barnwell, of the Elephant, Rhinoceros, Buffalo, Deer, Horse.

By Will. Mandell, M.A. (Qu.): A letter of Sir Isaac Newton to Mr Acland of Geneva was read. The following is a copy of the letter.

Vir celeberrime,

Gratias tibi debeo quam maximas quod schema experimenti quo lux in colores primitivos et immutabiles separatur, emendasti, et longe elegantius reddidisti quam prius. Sed et me plurimum obligasti quod schema in Caminâ æneâ incisum et inter imprimendum obtritum refici curasti, ut impressio libri elegantior redderetur. Gratias igitur reddo tibi quam amplissimos. Quod inventa mea de natura lucis et colorum viris summis Domino Cardinali Polignac et Domino Abbati Bignon non displiceant, valdè gaudeo. Utinam hæc vestratibus non minus placerent, quam elegantissimæ vestræ et perfectissimæ delineatæ picturæ nostratibus placuerunt. Ut Deus te liberet a doloribus capitis et salvum conservet ardentissime precatur

Servus tuus humillimus et obsequentissimus

Dabam Londini ISAACUS NEWTON.
22 Oct. 1722.

Celeberrimo viro Dno. Acland.

By Professor Sedgwick: On the geology of Teesdale.

March 15, 1824.

By Will. Mandell, M.A. (Qu.): Description of a self-regulating lamp.

By Geo. Biddell Airy, B.A. (Trin.): On the figure of the planet Saturn. *Trans.* II. 203—216.

By Professor Sedgwick: On the geology of Teesdale (continued).

March 29, 1824.

By Will. Mandell, M.A. (Qu.): On a means of protecting locks from the insertion of skeleton keys.

By G. Harvey, F.R.S.E., M.G.S.V.: On the fogs of the Polar seas.

By Professor Sedgwick: On the geology of Teesdale (concluded). *Trans.* II. 139—195.

May 3, 1824.

By Cha. Babbage, M.A. (Pet.): On the determination of the general terms of a new class of infinite series. (Read by Mr Peacock.) *Trans.* II. 217—225.

By Geo. Biddell Airy, B.A. (Trin.): On the construction of a new achromatic telescope.

May 17, 1824.

By Joh. Hogg, B.A. (Pet.): On two petrifying springs in the neighbourhood of Norton in the County of Durham. (Read by Professor Henslow.)

By Geo. Biddell Airy, B.A. (Trin.): On the principle and construction of the achromatic eyepieces of telescopes, and on the achromatism of microscopes. *Trans.* II. 227—252.

May 24, 1824.

By Professor Haviland, *President*: On the cases of secondary smallpox, and of smallpox after vaccination, which have occurred in Cambridge during the last year.

By Professor Farish: On a method of obviating the inconveniences arising from the expansion and contraction of the iron in iron bridges.

November 15, 1824.

By Professor Cumming: On the use of gold leaf in the detection of galvanism.

By Will. Whewell, M.A. (Trin.): On the principles of dynamics.

November 29, 1824.

By Professor Cumming: On the history of electro-magnetism.

December 13, 1824.

By Professor Farish: On the construction of the cogs of wheels.

Professor Farish likewise exhibited to the Society the action of wheels in the form of involutes of circles upon each other as an illustration of the subject of his paper.

February 21, 1825.

By Professor Cumming: On the conversion of iron into plumbago by the action of sea-water. *Trans.* II. 441—443.

By Geo. Biddell Airy, B.A. (Trin.): On a peculiar defect of his eyes producing distortion of images, and on the means of correcting it. *Trans.* II. 267—271.

By Professor Sedgwick: On the essential distinction between alluvial and diluvial deposits. *Annals of Philosophy*, X. 1825, pp. 18—37.

March 7, 1825.

By Will. Whewell, M.A. (Trin.): On a general method of converting rectilineal figures into others which are equivalent, such as squares, etc.

By Professor Sedgwick: On alluvial and diluvial deposits (continued).

March 21, 1825.

By Jos. Power, M.A. (Cla.): A general demonstration of the principle of virtual velocities. *Trans.* II. 273—276.

April 18, 1825.

By Professor Farish: On the construction of the cogs of wheels (concluded).

May 2, 1825.

By Geo. Biddell Airy, B.A. (Trin.): On the generation of curves by the rolling of one curve upon another, and on the formation of the curves of the teeth of wheels which may work in each other with perfect uniformity of action. *Trans.* II. 277—286.

By Professor Sedgwick: A portion of a paper on the geology of the Yorkshire coast, a section of which was exhibited to the Society.

By Will. Whewell, M.A. (Trin.): Exhibition of drawings of the appearances presented by the spokes of wheels in motion when seen through parallel bars, and which consist of a series of quadratures.

May 16, 1825.

By Ja. Alderson, B.A. (Pemb.): An account, with measurements, of an enormous whale cast upon the coast of Holderness. (Read by Professor Cumming.) *Trans.* II. 253—266.

By Professor Sedgwick: On the geology of the Yorkshire coast (concluded).

By Will. Whewell, M.A. (Trin.): On the classification of crystalline forms, particularly with reference to the systems of Weiss of Berlin and Mohs of Freyberg.

November 14, 1825.

By Ri. Wellesley Rothman, B.A. (Trin.): On the discrepancies between the magnetic intensities at different places on the earth's surface, as determined by observation, and by a formula partly empirical and partly theoretical of Horsteen and Barlow. *Trans.* II. 445.

By Geo. Biddell Airy, B.A. (Trin.): On the connection of impact and pressure, and the explanation of their effects on the same principles.

By Leonard Jenyns, M.A. (Joh.): On the ornithology of Cambridgeshire (read by Professor Henslow).

November 28, 1825.

By Leonard Jenyns, M.A. (Joh.): On the ornithology of Cambridgeshire (concluded). *Trans.* II. 287—324.

December 12, 1825.

By Ch. Babbage, M.A. (Pet.): On the principles of mathematical notation (read by Mr Peacock).

February 13, 1826.

By Will. Whewell, M.A. (Trin.): On the notation of crystallography. *Trans.* II. 427—439.

By Professor Farish: Explanation of a method of correcting the errors from the near position of a meridian mark.

February 27, 1826.

By Will. Woodall, M.A. (Pemb.): On a method of finding the meridian line.

By Professor Farish: Supplement to a paper read at the last meeting.

By Geo. Peacock, M.A. (Trin.): On Greek arithmetical notation.

March 13, 1826.

By Will. Hen. Wayne, M.A. (Pet.): On beds containing fossil bones intermixed with clay and gravel. A letter read, with observations, by Professor Sedgwick.

By Geo. Peacock, M.A. (Trin.): On Greek arithmetic (concluded).

April 10, 1826.

By Geo. Peacock, M.A. (Trin.): On the origin of Arabic Numerals, and the date of their introduction into Europe.

April 24, 1826.

By Will. Whewell, M.A. (Trin.): On a new method of Perspective; particularly for objects comprehending a large vertical and a small horizontal space.

By Geo. Peacock, M.A. (Trin.): On the origin of Arabic Numerals, etc. (concluded).

May 8, 1826.

By Geo. Biddell Airy, M.A. (Trin.): Observations on the Mécanique Celeste of Laplace, Book III., with some remarks on the objections of Mr Ivory. *Trans.* II. 379—390.

By Professor Sedgwick: On the Geology of the Isle of Wight.

November 13, 1826.

By Professor Sedgwick: Exhibition of a pair of large fossil horns, of some species of the genus *Bos*, found near Walton in Essex.

By Will. Whewell, M.A. (Trin.): On the classification of crystalline combinations, and the canons of derivation by which their laws may be investigated. *Trans.* II. 391—425.

November 27, 1826.

By Geo. Biddell Airy, M.A. (Trin.): On the motion of a pendulum disturbed by any small force, and on the application of this method to the theory of escapements. *Trans.* III. 105—128.

December 11, 1826.

By Geo. Peacock, M.A. (Trin.): On the numerals of the South American languages.

After the meeting Professor Airy gave an account of the construction and application of the steam-engine in the mines of Cornwall.

February 26, 1827.

By Professor Airy: On the mathematical theory of the Rainbow.

After the meeting Professor Henslow gave an account of the structure of the capsules of mosses, illustrated by coloured drawings.

March 12, 1827.

By Geo. Peacock, M.A. (Trin.): On the discoveries recently made on the subject of the Egyptian Hieroglyphics.

March 26, 1827.

By Professor Henslow: On the specific identity of the Cowslip, Oxlip, and Primrose.

By Will. Whewell, M.A. (Trin.): Note on the perspective projection of objects on a horizontal plane.

After the meeting Professor Cumming gave an account of the different forms of the Galvanometer, and of the discoveries recently made in Electrodynamics.

April 30, 1827.

By Will. Sutcliffe, M.A. (Trin.): On the application of mathematics to Political Economy, and the effects of a partial Tithe.

By Will. Whewell, M.A. (Trin.): On the Perspective of Panoramas.

After the meeting Professor Sedgwick exhibited a large pair of horns of [some species of the genus *Bos*] found near Walton in Essex; and an *Ichthyosaurus*, found at Lyme; on which he offered some observations.

May 14, 1827.

By Will. Sutcliffe, M.A. (Trin.): On the application of mathematics to Political Economy, etc. (concluded).

By Professor Airy: On the defects of the eye-pieces of telescopes.

After the meeting Professor Sedgwick gave an account of the peculiarities of the Coal Strata in the neighbourhood of Whitehaven: and George Noakes (æt. 7), a boy remarkable for his powers of calculation, was examined by several members of the Society.

May 21, 1827.

By R. M. Fawcett: On the use of Iodine in cases of Paralysis.

By Professor Airy: On the observation of eye-glasses depending upon their spherical figure, and on the periscopic Panorama. *Trans.* III. 1—63.

After the meeting Mr Peacock gave an account of the discoveries recently made in Hieroglyphics.

November 12, 1827.

By Tho. Jarrett, B.A. (Cath.): On Algebraical Notation. *Trans.* III. 65—103.

By Will. Whewell, M.A. (Trin.): On the History and Principles of Chemical Nomenclature and Notation, with suggestions of some alterations in the Notation hitherto in use.

By Will. Mandell, B.D. (Qu.): Exhibition of a piece of breccia, supposed to be a fragment of a Roman quern or hand-mill, found on the Hills Road.

November 26, 1827.

Professor Sedgwick read a letter from Mr Ri. Tho. Lowe, concerning certain petrifactions, apparently of vegetable origin, which are found in the Island of Madeira.

By Professor Henslow: An account of the application of the chloraret of lime to the purpose of disinfecting and neutralizing putrid and noxious substances.

December 10, 1827.

Dr Fre. Thackeray presented a sword of the sword-fish, and read some observations on the bones of the head, and especially those which seem to belong to its olfactory system.

By Leonard Jenyns, M.A. (Joh.): On the monstrous prolongations of teeth, etc., which have been observed in different animals, particularly the teeth of a rabbit and the bill of a rook which exist in the Collection of the Society; and on the circumstances by which such deformities have been observed to be accompanied.

February 17, 1828.

By Alex. Ch. Louis D'Arblay, M.A. (Chr.): Remarks on a pamphlet by Messrs Swinburne and Tylecot of St John's College, concerning the proofs of the Binomial Theorem, and especially that of Euler.

After the meeting Mr Peacock gave an account of the representations occurring in Egyptian monuments of the deities of that country, and of the funeral rituals.

March 3, 1828.

By Alex. Thomson (Joh.): On a mode of obtaining exact measures of the cranium.

By Will. Whewell, M.A. (Trin.): On the different systems of mineralogical classification.

After the meeting Professor Sedgwick gave an account of the geological structure of Scotland, as collected from the observations made by himself and Mr Murchison during the preceding summer.

March 17, 1828.

By Tho. Jarrett, M.A. (Cath.): On the development of Polynomials.

By Will. Whewell, M.A. (Trin.): On the different systems of mineralogical classification (concluded).

After the meeting Hen. Coddington, M.A. (Trin.) gave an account of the experiments on vibrations and nodal lines of Chladni, Savart, on the construction of organ-pipes, etc.

April 21, 1828.

By Will. Whewell, M.A. (Trin.): On mineralogical nomenclature.

By Temple Chevallier, M.A. (Pem.): On certain properties of numbers.

By Rob. Willis, B.A. (Cai.): On the pressure of the air between two discs when affected by a stream of air passing through a tube perforating one of the discs. *Trans.* III. 129—140.

After the meeting Mr Willis exhibited various experiments illustrative of the laws of pressure described in his memoir.

May 5, 1828.

By Tho. Jarrett, M.A. (Cath.): On the arithmetic of lines.

By Professor Whewell: On mineralogical classification (concluded).

After the meeting Professor Haviland gave an account of the nature and use of the stethoscope.

May 19, 1828.

By Thos. Jarrett, M.A. (Cath.): On two theorems useful in the integration of certain functions.

By Joh. Will. Lubbock, M.A. (Trin.): On the calculation of Annuities, and some theorems in the doctrine of chances. *Trans.* III. 141—154.

November 10, 1828.

By Ja. Challis, M.A. (Trin.): On the Law of Distances applied to the Satellites.

After the meeting Professor Whewell gave a lecture on the granite veins of Cornwall.

November 24, 1828.

By Professor Airy: On the Longitude of Cambridge. *Trans.* III. 155—170.

By Rob. Willis, M.A., Gonv. and Cai. Coll.: On the vowel sounds.

After the meeting Mr Willis exhibited experiments illustrative of his doctrines.

December 8, 1828.

By Joh. Warren, M.A. (Jes.): On the doctrine of impossible quantities, and their geometrical representation, and on the proof that every equation of n dimensions has n roots.

By Fre. Thackeray, M.D. (Emm.): On the case of Ann Carter, a young woman at Stapleford, said to be a trance.

By Ja. Challis, M.A. (Trin.): On the Law of Distances, etc. (concluded). *Trans.* III. 171—183.

After the meeting Mr Leonard Jenyns gave an account, illustrated by drawings, of the comparative anatomy of Birds and Mammals, and of several particulars respecting the former Class.

March 2, 1829.

By Pierce Morton, B.A. (Trin.): On the focus of a conic section. *Trans.* III. 185—190.

By Professor Whewell: On the application of mathematical reasoning to certain theories of Political Economy.

After the meeting Professor Whewell gave an account of various contrivances employed in dipping needles, and of some suggested improvements.

March 16, 1829.

By Professor Whewell: On the application of mathematical reasoning, etc. (concluded). *Trans.* III. 191—230.

By Rob. Willis, M.A. (Gonv. & Cai.): On the theory of the sounds of pipes as relating to their vowel quality (concluded from 24 Nov. 1828). *Trans.* III. 231—268.

After the meeting Mr Willis exhibited experiments illustrative of the influence of the length of the pipe on the vibrations of the reed, and of the different ways in which the vowel sounds may be produced.

March 30, 1829.

By Ja. Challis, M.A. (Trin.): Abstract of a memoir on the vibrations of elastic fluids. *Trans.* III. 269—320.

By Joh. Will. Lubbock, M.A. (Trin.): On the tables of the chances of life, and on the value of annuities. *Trans.* III. 321—341.

After the meeting Professor Henslow gave an account of the organization and classification of ferns, illustrated by drawings.

May 4, 1829.

By Professor Whewell: On the mineralogical systems proposed by Nordenskiold, Bernsdorff, Kefersheim, and Naumann.

After the meeting Mr Leonard Jenyns gave an account of the construction, properties, and mode of growth, of feathers.

May 18, 1829.

By Will. Hallows Miller, M.A. (Joh.): On caustics formed by successive reflexions at a spherical surface.

By Rob. Willis, M.A. (Gonv. & Cai.): On the mechanism of the human voice. *Trans.* IV. 323—352.

After the meeting Mr Willis exhibited various experiments and models, and explained the action of the organs of voice.

November 16, 1829.

By Professor Airy: On a correction of the length of a pendulum consisting of a wire and ball. *Trans.* III. 355—360.

By Professor Whewell: On the causes and characters of Pointed Architecture.

After the meeting Professor Whewell described the kinds of vaulting employed in German churches, with their history; illustrating his account with models.

November 30, 1829.

By Ri. Wellesley Rothman, M.A. (Trin.): On an observation of a solstice at Alexandria recorded by Strabo. *Trans.* III. 361—363.

By Professor Whewell: On Pointed Architecture (concluded).

By Will. Hallows Miller, M.A. (Joh.): On the forms and angles of certain crystals. *Trans.* III. 365—367.

After the meeting Professor Sedgwick gave an account of the geology and structure of the Alps, illustrated by a section from the plains of Bavaria to those of Trieste.

December 14, 1829.

By Professor Airy: On the mathematical conditions of continued motion. *Trans.* III. 369—372.

By Ch. Pleydell Neale Wilton, M.A. (Joh.): On the geology of the shore of the Severn in the Parish of Awre in Gloucestershire.

After the meeting Mr Leonard Jenyns gave an account of the circumstances connected with the migration of Birds.

February 22, 1830.

By Ja. Challis, M.A. (Trin.): On the integration of the equations of motion of fluids; and on the application of this to the solution of various problems. *Trans.* III. 383—416.

By Leonard Jenyns, M.A. (Joh.): On the Natter-Jack of Pennant, with a list of Reptiles found in Cambridgeshire. *Trans.* III. 373—381.

After the meeting Professor Henslow explained the discoveries of M. Dutrochet on Endosmose and Exosmose.

March 8, 1830.

By Ch. Pleydell Neale Wilton, M.A. (Joh.): Account of a visit to Mount Wingen, the burning mountain of Australia.

By Hen. Coddington, M.A. (Trin.): On the construction of a microscope invented by him, which he exhibited to the Society.

After the meeting Professor Airy gave an account, illustrated by models, of the instruments which have been used to measure altitudes: viz. the Zenith Sector, the Quadrant, the Refracting Circle, the large Declination Circles of Troughton, and the Circles of Reichenbach.

March 22, 1830.

By Will. Hallows Miller, M.A. (Joh.): On the measurement of the angles of certain crystals which occur in the slags of furnaces. *Trans.* III. 417—420.

By Hen. Coddington, M.A. (Trin.): On the advantages of a microscope of a new construction. *Trans.* III. 421—428.

By Hugh Ker Cantrien, B.A. (Trin.): On the Calculus of Variations.

Mr Willis gave an account, illustrated by models and drawings, of the construction and muscles of the tongue, palate, and pharynx, and of the mode in which these operate in the production of vowel and modulated sounds.

April 26, 1830.

By Leon. Jenyns, M.A. (Joh.): On the late severe winter.

By Hen. Coddington, M.A. (Trin.): On his new-invented microscope.

By Professor Whewell: On the proof of the first law of motion.

After the meeting Professor Whewell gave an account of Göthe's objections to the Newtonian theory of Optics, and of the doctrine proposed by that author.

May 10, 1830.

By Tho. Chevallier, M.D.: On the anatomy and physiology of the ear.

After the meeting Professor Cumming explained the construction and use of the areometer of Professor Leslie, and its resemblance to the stereometer of Captain Say; and the construction of an instrument for measuring the whole quantity of sunshine which operates during any given time.

May 24, 1830.

By Professor Airy: On the peculiar form of the rings produced by a ray circularly polarized, and on the calculation of the intensity of light belonging to this and other cases.

By Will. Webster Fisher (Down.): On the appendages to organs as provisionary to the modifications of the functions.

By Rob. Murphy, B.A. (Gonv. and Cai.): On the general properties of definite integrals, and on the equation of Riccati. *Trans.* III. 429—443.

By Hen. Coddington, M.A. (Trin.): A further explanation of his microscope.

After the meeting Mr Willis exhibited and explained an instrument for making orthographical projections of objects.

November 15, 1830.

By Aug. De Morgan, B.A. (Trin.), Professor of Mathematics in London University: On the Equation of Curves of the second degree. *Trans.* IV. 71—78.

By Will. Okes, M.A. (Gonv. and Cai.): On the Wourali poison used by the Maconshi Indians; a blow-pipe, quiver, and arrows were exhibited.

By Professor Cumming: A communication from Mr Edwards on a substance resembling cannel coal, found in digging a canal near Norwich.

By Ri. Tho. Lowe, B.A. (Chr.): On the Natural History of Madeira.

After the meeting Professor Whewell gave an account of a method of constructing cross vaults without boarded centering, revived and described by M. de Lassaulx of Coblentz.

November 29, 1830.

By Ri. Tho. Lowe, B.A. (Chr.): On the Natural History of Madeira (concluded). *Trans.* IV. 1—70.

By Professor Whewell: Rules for the selection and employment of symbols of mathematical quantity.

After the meeting Mr Leonard Jenyns gave an account, illustrated by drawings, of the quinary system of Natural History proposed by Mr M^cLeay.

December 13, 1830.

By Professor Whewell: Rules for the selection, etc. (concluded).

By Professor Henslow: On the mode of reproduction of the *Chara.*

After the meeting Professor Henslow made some observations on tall ferns, exhibiting a specimen of a stalk.

A machine was exhibited invented by Professor Airy for the purpose of exhibiting the mode of propagation of undulations along a line of particles.

February 21, 1831.

By Professor Airy: On the nature of the two rays formed by the double refraction of quartz. *Trans.* IV. 79—123.

After the meeting Professor Airy exhibited a machine for illustrating the nature of the undulations supposed in circular polarization; an instrument for exhibiting the rings, spirals, etc. produced by double refraction; and an instrument for exhibiting the same phenomena by means of the light produced by the combustion of lime in oxygen.

March 7, 1831.

By Rob. Murphy, B.A. (Gonv. and Cai.): On the general solution of equations. *Trans.* IV. 125—153.

After the meeting Mr Willis exhibited a series of experiments on the transverse and longitudinal vibrations of strings, membranes, and solids, illustrative of the researches of M. Savart.

March 21, 1831.

By Will. Hallows Miller, M.A. (Joh.): On the elimination of the time from the differential equations of the motion of a point, whether affected by a resisting medium, or by any disturbing forces.

By the same: On measurements of the angles of certain artificial crystals.

After the meeting Mr Willis exhibited and explained a machine constructed for the purpose of illustrating the motion of the particles of any medium in which undulations are propagated.

April 18, 1831.

By Professor Whewell: On the mathematical exposition of some of the leading doctrines of Mr Ricardo's *Principles of Political Economy and Taxation.*

By Professor Airy: Notice of an apparatus constructed under his direction by Mr Dollond, and of the phenomena of elliptically polarized light exhibited by means of the apparatus. *Trans.* IV. 199—208.

After the meeting Professor Henslow exhibited a series of appearances produced by two wheels revolving one behind the other.

May 9, 1831.

By Ch. Pritchard, B.A. (Joh.): A method of simplifying the demonstration of the two principal theorems respecting the figure of the earth considered as heterogeneous.

By Professor Whewell: On the mathematical exposition, etc. (concluded). *Trans.* IV. 155—198.

After the meeting Mr Willis exhibited apparatus illustrating the nature of sound, and the vibrations which produce it, especially an instrument which he calls a Lyophone.

May 16, 1831.

By Ja. Francis Stephens: Description of *Chiasognathus grantii*, a new Lucanideous insect forming the type of an undescribed genus, together with some brief remarks upon its structure and affinities. *Trans.* IV. 209—216.

By Professor Clark: On a monster of the kind called semidouble. *Trans.* IV. 219—255.

After the meeting Mr Willis exhibited Mr Trevelyan's experiment on the rocking of a bar of hot brass placed upon a plate of cold lead.

Mr Leonard Jenyns gave an account of the application of the quinary system of Mr McLeay to the classification of Birds.

November 14, 1831.

By Professor Airy: On some new circumstances in the phenomena of Newton's rings. *Trans.* IV. 279—288.

By Professor Henslow: On a hybrid plant between *Digitalis purpurea* and *D. lutea.*

After the meeting Professor Sedgwick gave an account, illustrated by sections, of the geological structure of Caernarvonshire.

November 28, 1831.

By Leonard Jenyns, M.A. (Joh.): A monograph of the British species of bivalve mollusca belonging to the genera *Cyclas* and *Pisidium.* *Trans.* IV. 289—312.

By Sam. Earnshaw, B.A. (Joh.): On the integration of the general linear differential equation of the *n*th order, and the general equation of differences with constant coefficients.

After the meeting Professor Whewell gave an account of the theories of evaporation, the use of Daniel's hygrometer, and the formation of clouds.

December 12, 1831.

By Professor Cumming: Exhibition of a calculus found in the intestines of a horse, with remarks.

By Professor Henslow: On a hybrid *Digitalis* (concluded). *Trans.* IV. 257--278.

After the meeting Mr C. Jenyns gave an account, illustrated by drawings, of the rules of the perspective of shadows, and explained the use of the centrolinead.

March 5, 1832.

By Professor Airy: On a new analyser of light polarized in a peculiar manner. *Trans.* IV. 313--322.

By Rob. Murphy, B.A. (Gonv. and Cai.): On an inverse calculus of definite integrals. (*Trans.* IV. 353—408.)

After the meeting Professor Airy exhibited an apparatus illustrative of some of the phenomena referred to in his paper.

Professor Henslow gave a lecture, illustrated by specimens and drawings, on the age of trees.

March 19, 1832.

By Professor Airy: On the phenomena of Newton's rings when formed between two transparent substances of different refractive powers. *Trans.* IV. 409--424.

By Will. Brett, M.A. (Corp.): On the phenomena of double stars.

After the meeting Mr Whewell gave an account, illustrated by diagrams, of the forms and course of the cotidal lines according to the causes which influence them, and according to the observations made in different places.

April 2, 1832.

By J[ohn] P[rentice] Henslow: On the habits of two hybrid pheasants presented by him to the Society.

By B. Bushell: On the anatomy of the same birds.

By Will. Brett, M.A. (Corp.): On the theory of stars of variable brightness.

May 7, 1832.

By Sir Joh. Fre. Will. Herschel, M.A. (Joh.): Description of a machine for solving equations. *Trans.* IV. 425—440.

By Will. Holt Yates, M.B. (Joh.): Account of the magnetic mountain of Sipylus near Magnesia.

After the meeting Professor Sedgwick gave an account, illustrated by maps and sections, of the Physical Geography and History of the Fens of Cambridgeshire.

May 21, 1832.

By Sir Joh. Fre. Will. Herschel, M.A. (Joh.): Description of a machine, etc. (concluded).

By Rob. Willis, M.A. (Gonv. and Cai.): On the ventricles of the larynx.

By Professor Henslow: On a monstrosity of *Reseda*. *Trans.* V. 95--100.

After the meeting Mr Willis gave a lecture, illustrated by experiments, on various phenomena of sound.

June 4, 1832.

By Joh. Hogg, M.A. (Pet.): On the classical plants of Sicily.
By Professor Henslow: Exhibition of a drawing representing the construction of *Reseda*, in illustration of his former paper.
By Professor Clark: Exhibition of a semi-double fœtus of a pig, with explanation.
By Professor Cumming: On Mr Faraday's recent discoveries in magneto-electricity, with illustrative experiments.

November 12, 1832.

By Geo. Green: Mathematical investigations concerning the laws of the equilibrium of fluids analogous to the electric fluid; with other similar researches. *Trans.* v. 1—63.
By Aug. De Morgan, B.A. (Trin.): On the general equation of surfaces of the second degree. *Trans.* v. 77—94.
After the meeting Professor Henslow gave an account, illustrated by various drawings and diagrams, of various observations of Geology and Natural History made by him during a residence at Weymouth during a portion of the summer.

November 26, 1832.

By Rob. Murphy, M.A. (Gonv. and Cai.): On an elimination between an indefinite number of unknown quantities. *Trans.* v. 65—75.
By Will. Whewell, M.A. (Trin.): On the architecture of Normandy.
After the meeting Mr Ch. Brooke (Joh.) gave an account of the history and recent improvements in Lithotripsy, illustrated by the exhibition of the instruments used, and by several drawings.

December 10, 1832.

By Will. Whewell, M.A. (Trin.): On the architecture of Normandy (continued).
After the meeting Mr Sims gave an account of the various methods of engine-dividing, and of original dividing practised with regard to graduated instruments; and explained particularly the method of original dividing invented by Mr Troughton, and recently applied by Mr Sims to the division of the mural circle of the Observatory. This explanation was illustrated by models and apparatus.

February 25, 1833.

By Will. Whewell, M.A. (Trin.): On the architecture of Normandy (continued)
After the meeting Professor Airy gave an account, illustrated by models and diagrams, of his researches concerning the mass of Jupiter by means of observations of the Fourth Satellite.

March 11, 1833.

By the Marchese Spineto: An examination of the grounds of Sir Isaac Newton's system of chronology.
After the meeting Professor Sedgwick gave an account, illustrated by representations of sections, of the Geology of North Wales.

March 25, 1833.

By Jos. Power, M.A. (Trin. Hall): On the effect of wind on the barometer.
By Professor Clark: On an unusual situation of the origin of the internal mammary artery, with a drawing and explanation.
After the meeting Professor Henslow gave an account, illustrated by diagrams, of a method of classifying and designating colours, particularly with reference to their use in natural-historical descriptions.

April 22, 1833.

By Professor Miller: On lines produced in the spectrum by the vapour of Bromine, Iodine, and Euchlorine. *Phil. Mag.* 1833, i. 381.

By Will. Whewell, M.A. (Trin.): On the architecture of Normandy (concluded).

After the meeting Mr Whewell explained some of the difficulties which had attended his researches concerning co-tidal lines.

May 6, 1833.

By Mr Millsom: A description of the anatomy of a hybrid animal—a lion-tiger (communicated by Dr Haviland).

By Geo. Green: A memoir on the exterior and interior attractions of ellipsoids (communicated by Sir Edw. Tho. French Bromhead, Bart., M.A. Gonv. and Cai.). *Trans.* v. 395—429.

By the Marchese Spineto: On an insect which appears in the Egyptian Hieroglyphics.

By Professor Airy: On diffraction. *Trans.* v. 101—111.

May 20, 1833.

By Will. Hopkins, M.A. (Pet.): On the position of the nodes of the vibration of the air in tubes. *Trans.* v. 231—270.

Mr Hopkins also exhibited experiments illustrating the interference of the vibrations of the air.

November 11, 1833.

By Rob. Murphy, M.A. (Gonv. and Cai.): A second memoir on the inverse method of definite integrals. *Trans.* v. 113--148.

By Professor Airy: An account of various observations made on the Aurora Borealis of September 17 and October 12. *Phil. Mag.* 1833, ii. 461.

November 25, 1833.

By Professor Henslow: Observations on a beetle found in a block of mahogany presented to the Society.

By Ri. Tho. Lowe, M.A. (Chr.): Description of a molluscous animal of the genus *Umbrella*, with a drawing and remarks.

By Will. Hopkins, M.A. (Pet.): On the geology of Derbyshire, illustrated by maps and sections. *Phil. Mag.* 1834, i. 66.

December 9, 1833.

By Hen. Moseley, B.A. (Joh.), Professor of Natural Philosophy in King's Coll. Lond.: On the general conditions of the equilibrium of a system of variable form; and on the theory of equilibrium, fall, and settlement, of the arch. *Trans.* v. 293—313.

By Professor Farish: On the appearance of a meteor, or falling star, of great splendour, observed by him at a quarter before seven o'clock, on September 26 (he being near Magdalene College).

By Professor Sedgwick: On the geology of Charnwood Forest, illustrated by maps and sections. *Phil. Mag.* 1834, i. 68.

February 17, 1834.

By Fra. Lunn, M.A. (Joh.): On a specimen of *Proteus anguinus*, presented by him to the Society.

By Professor Miller: Some optical observations on lines in the vapour of Iodine, Bromine, and Perchloride of Chrome. *Phil. Mag.* 1834, i. 312.

By Will. Whewell, M.A. (Trin.): On the nature of the truth of the Laws of Motion. *Trans.* v. 149—172.

March 3, 1834.

By Ja. Challis, M.A. (Trin.): On the motion of fluids. *Trans.* v. 173—203.
By Temple Chevallier, B.D. (Cath. H.): On the polarisation of the light of the atmosphere. *Phil. Mag.* 1834, i. 312.
By Professor Miller: Notice of experiments on the Perchloride of Chrome.

March 17, 1834.

By Jos. Power, M.A. (Trin. Hall): On the theory of Exosmose and Endosmose. *Trans.* v. 205—229.
By Professor Henslow: On Braun's speculations concerning the arrangement of the scales on fir-cones, with additional remarks.
By Professor Airy: On the polarisation of light by the sky, and by rough bodies. *Phil. Mag.* 1834, i. 313.

April 14, 1834.

By Professor Airy: On the latitude of the Cambridge Observatory, as determined by means of the mural circle. *Trans.* v. 271—281.
By Will. Whewell, M.A. (Trin.): On Sir J. Herschel's hypothesis concerning the absorption of light by coloured media. *Phil. Mag.* 1834, i. 463.

April 28, 1834.

By Professor Miller: On the axes of crystals. *Phil. Mag.* 1834, i. 463.
By Sam. Earnshaw, B.A. (Joh.): On the laws of motion. Ibid.
After the meeting Mr Willis explained a machine of his construction for jointing together the bones of skeletons.

May 12, 1834.

By Aug. De Morgan, B.A. (Trin.): An attempt to shew that the principles of the Differential Calculus may be established without assuming the forms of any expansion (in a letter to Mr Peacock).
After the meeting Professor Miller exhibited and explained an instrument for taking the specific gravities of bodies.
[By Rob. Willis, M.A. (Gonv. and Cai.): Exhibition and explanation of an instrument constructed by himself, which he proposes to call an Orthograph.
By Will. Webster Fisher, M.B. (Down.): On the origin of Tubercular diseases.]

November 10, 1834.

By Ri. Tho. Lowe, M.A. (Chr.): Descriptions of six new or rare species of fish from Madeira, with drawings. *Trans.* VI. 195—201.
By Will. Whewell, M.A. (Trin.): Observations of the tides made from June 7 to June 22, 1834, at the coastguard stations; with some observations on the mode of discussing them.

November 24, 1834.

By Professor Airy: On the rings produced by viewing the image of a star through an object-glass of circular aperture. *Trans.* v. 283—291.
By the same: On the longitude of the Cambridge Observatory, as compared with the result of the Trigonometrical Survey.
By Ri. Stevenson, B.A. (Trin.): On the establishment of propositions by the infinitesimal method combined with the doctrine of projections.
By Professor Sedgwick: On the geology of Cambridge. *Phil. Mag.* 1835, i. 74.

December 8, 1834.

By Professor Miller: On the position of the optical axes of crystals. *Trans.* v. 431—438.

By Professor Henslow: On the Green Sand at Haslingfield, Barton, etc.

By the same: On the age of trees, as determined by their size.

By Professor Airy: On the echo from the open end of a tall chimney.

[By Professor Cumming: A statement of Melloni's discoveries on the transmission of heat by radiation.]

March 2, 1835.

By Rob. Murphy, M.A. (Gonv. and Cai.): On the inverse method of Definite Integrals. *Trans.* v. 315—393.

By Ri. Stevenson, B.A. (Trin.): On the solution of some problems connected with the theory of straight lines and planes by a new symmetrical method of coordinates.

By Will. Hopkins, M.A. (Pet.): On Physical Geology. *Trans.* VI. 1- -84.

March 16, 1835.

By Will. Webster Fisher, M.B. (Down.): On the nature, structure, and changes, of tubercles, illustrated by coloured drawings. *Phil. Mag.* 1835, i. 395.

After the meeting Mr Willis gave an account, illustrated by drawings and models, of the progress of Gothic Architecture, and especially of the formation of tracery. Ibid.

March 30, 1835.

By Aug. de Morgan, B.A. (Trin.): On the theorem of M. Abel relative to the algebraical expression of the roots of equations which are connected by the law of periodic functions.

By Will. Whewell, M.A. (Trin.): Exhibition and explanation of an Anemometer of a new construction; with a statement of the use which might be made of observations made by means of it.

May 4, 1835.

By Professor Airy: An account of results recently obtained at the Observatory with respect to: (1) the obliquity of the ecliptic; (2) the mass of Jupiter; (3) Jupiter's time of rotation.

By Will. Whewell, M.A. (Trin.): On the results of the Tidal Observations of the Coast Guard of June, 1834; and on those intended to be made in June, 1835.

May 18, 1835.

By Archib. Smith (Trin.): A communication containing the eliminations by which the equation of the wave surface in Fresnel's theory of undulations is determined in a manner more simple than in previous investigations of other authors on the same subject (read by Professor Airy). *Trans.* VI. 85—89.

By Will. Whewell, M.A. (Trin.): An extract of a letter from Professor Schumacher, stating that Messrs Beer and Mödler had found the time of Jupiter's rotation to be $9^h\ 55^m\ 26^s, 5$; and that M. Bessul had made a long series of observations which give the mass of Jupiter nearly identical with those of Professor Airy.

By Will. Webster Fisher, M.B. (Down.): On tubercles (continued).

June 1, 1835.

By Rob. Willis, M.A. (Gonv. and Cai.): An account, illustrated by models, of the progress of decorative construction in vaults. *Phil. Mag.* 1835, i. 71.

November 16, 1835.

By Rob. Murphy, M.A. (Gonv. and Cai.): On the resolution of equations of finite differences. *Trans.* VI. 91—106.

Extracts of letters written by Sir J. F. W. Herschel, M.A. (Joh.) from the Cape of Good Hope, on meteorological observations made by him there.

Extracts of letters from Ch. Rob. Darwin, B.A. (Chr.) containing accounts of the geology of certain parts of the Andes and S. America.

November 30, 1835.

By Will. Wallace, F.R.S., Prof. of Mathematics, Edinburgh (Hon. Member): On a geodetical problem. *Trans.* VI. 107—140.

By Professor Airy: On a supposed analysis of the spectrum by Sir D. Brewster.

December 14, 1835.

By Ri. Potter (Qu.): On the explanation of the phenomena of the rainbow by the doctrine of interferences. *Trans.* VI. 141—152.

By Ch. Rob. Darwin, B.A. (Chr.): On viviparous lizards, and on red snow.

February 21, 1836.

By Phil. Kelland, B.A. (Qu.): On the dispersion of Light on the Undulatory Theory. *Trans.* VI. 153—184.

By Will. Whewell, M.A. (Trin.): On the Tides. *Phil. Mag.* 1836, i. 430.

March 7, 1836.

By Will. Whewell, M.A. (Trin.): On the recent discoveries of Professor Forbes and others respecting the polarisation of heat. *Phil. Mag.* 1836, i. 430.

After the meeting Mr Willis gave a lecture on the composition and resolution of the entablature in Egyptian and Grecian architecture. Ibid.

March 21, 1836.

By Sam. Earnshaw, M.A. (Joh.): On the solution of the equation of continuants of fluids in motion. *Trans.* VI. 203—233.

By Professor Miller: On the position of the axes of optical elasticity of certain crystals. *Trans.* VII. 209—215.

By Tho. Webster, M.A. (Trin.): On the connection of the periodical [motions] of the barometer with the changes of temperature; and on the relation of the accidental changes with the occasional changes.

April 18, 1836.

By Professor Sedgwick: An account of the system of formations inferior to the Carboniferous Series, as illustrated by his own researches in Wales, and those of Mr Murchison in the same country.

May 2, 1836.

By Geo. Biddell Airy, M.A. (Trin.), Astronomer Royal: On the intensity of light in the neighbourhood of the caustic. *Trans.* VI. 379—402.

By Will. Hopkins, M.A. (Pet.): On the agreement between his theoretical views of the elevatory geological forces, and the phenomena of faults, as observed by him in the strata of Derbyshire.

May 16, 1836.

By Aug. De Morgan, B.A. (Trin.): Sketch of a method of introducing discontinuous constants into the arithmetical expressions for infinite series. (In a letter to Mr Peacock.) *Trans.* VI. 185—193.

By Phil. Kelland, B.A. (Qu.): On the constitution of the atmosphere and the connexion of light and heat. *Trans.* VI. 235—288.

By Will. Hopkins, M.A. (Pet.): Observations on the temperature of mines, and the doctrine of central heat.

By Geo. Biddell Airy, M.A. (Trin.): Observations of temperature during the great Solar Eclipse of 15 May.

November 14, 1836.

No papers recorded.

November 28, 1836.

By Joh. Thompson Exley (Joh.): On the leading features of a new system of Physics.

By Professor Henslow: On various kinds of pebbles and agates, with conjectures respecting the origin of the bands of colour with which they are marked.

December 12, 1836.

By Sam. Earnshaw, M.A. (Joh.): On the appearance of light received on a screen after passing through an equilateral triangle placed behind the object-glass of a telescope. *Trans.* VI. 431—442.

By Ja. Jos. Sylvester (Joh.): On elimination, and the use of indeterminate constants.

By Will. Hopkins, M.A. (Pet.): On the formation of veins in Derbyshire.

February 13, 1837.

By Professor Challis: On the temperature of the higher regions of the atmosphere. *Trans.* VI. 443—455.

By Steph. Pet. Rigaud, Savilian Professor of Astronomy, Oxford: On the relative proportions of Land and Water. *Trans.* VI. 289—300.

By Phil. Kelland, B.A. (Qu.): On the transmission of light through crystallised media. *Trans.* VI. 323—352.

February 27, 1837.

By Joh. Warren, M.A. (Jes.): On the algebraical sign of the perpendicular from a given point upon a given line.

By Ch. Rob. Darwin, B.A. (Chr.): An account of fused sand-tubes found near the Rio Plata—which were exhibited, along with several other specimens of rocks.

By Will. Webster Fisher, M.B. (Down.): On a case of *Spina bifida*. *Phil. Mag.* 1837, i. 316.

March 13, 1837.

By Phil. Kelland, B.A. (Qu.): Supplement to his paper read 13 February.

By Sam. Earnshaw, M.A. (Joh.): On the laws of fluid motion.

By Hen. Joh. Hales Bond, M.D. (Corp.): A medical-statistical Report of Addenbrooke's Hospital for 1836. *Trans.* VI. 361—377.

By Will. Whewell, M.A. (Trin.): An account of the recent results of his researches respecting the Tides.

April 17, 1837.

By Leonard Jenyns, M.A. (Joh.): On the temperature of the month of March last past. *Phil. Mag.* 1837, i. 485.

By Rob. Willis, M.A. (Gonv. and Cai.): Exhibition and explanation of a Tabuloscriptive Engine. Ibid.

May 1, 1837.

By Art. Aug. Moore (Trin.): Solution of a difficulty in the analysis of Lagrange noticed by Sir W. Hamilton (read by Mr Peacock). *Trans.* VI. 317—322.

By Will. Whewell, M.A. (Trin.): On the results of his Anemometer for the first three months of 1837. *Trans.* VI. 301—315.

By Phil. Kelland, B.A. (Qu.): On the elasticity of the aether in crystals. *Trans.* VI. 353—360.

May 15, 1837.

By Geo. Green (Gonv. and Cai.): On the propagation of an undulation in heavy fluids in a canal of small depth and width. *Trans.* VI. 457—462.

By Will. Hopkins, M.A. (Pet.): On the refrigeration of the earth, and on the doctrine of internal fluidity of the earth.

By Hen. Moseley, M.A. (Joh.): On the theory of the equilibrium of bodies in contact. *Trans.* VI. 463—491.

[By Will. Webster Fisher, M.B. (Down.): On *Spina bifida*. *Phil. Mag.* 1837, i. 486.]

November 13, 1837.

By Car. Jeffreys, M.A. (Joh.): Exhibition and explanation of the Respirator invented by his brother.

By Professor Sedgwick: On the geology of Charnwood Forest and the neighbouring coalfields.

November 27, 1837.

By Geo. Green (Gonv. and Cai.): On the vibration of air. *Trans.* VI. 403—413.

By Will. Hopkins, M.A. (Pet.): On certain elementary principles of geological theory; and on Professor Babbage's speculations.

December 11, 1837.

By Geo. Green (Gonv. and Cai.): On the reflexion and refraction of light in non-crystallised media. *Trans.* VII. 1—24.

By Ri. Wellesley Rothman, M.A. (Trin.): On the observation of Halley's comet in 1836. *Trans.* VI. 493—506.

By Will. Hopkins, M.A. (Pet.): On Precession and Nutation, assuming the interior fluidity of the earth.

February 26, 1838.

By Dav. Tho. Ansted, B.A. (Jes.): On a new genus of fossil shells. *Trans.* VI. 415—422.

By Aug. De Morgan, B.A. (Trin.): On a question in the theory of probabilities. *Trans.* VI. 423—430.

By Dan. Cresswell, D.D. (Trin.): On the squaring of the circle.

March 12, 1838.

By Phil. Kelland, M.A. (Qu.): On molecular attraction. *Trans.* VII. 25—59.

By Professor Henslow: On plants brought by Mr Darwin from Keeling Island.

March 26, 1838.

By Geo. Biddell Airy, M.A. (Trin.): On the intensity of light in the neighbourhood of a caustic. *Trans.* VI. 379—402.

By Professor Challis: On the proper motions of the stars.

April 30, 1838.

By Ri. Potter, B.A. (Qu.): On a new correction in the construction of the double achromatic object-glass. *Trans.* VI. 553—564.

By Will. Hen. Trentham, M.A. (Joh.): On the expansion of a polynomial.

By Hen. Joh. Hayles Bond, M.D. (Corp.): Statistical report on Addenbrooke's Hospital for 1837. *Trans.* VI. 565—575.

By Pet. Bellinger Brodie, B.A. (Trin.): On the occurrence of recent land and fresh-water shells with bones of some extinct animals in the gravel near Cambridge; communicated by Professor Sedgwick. *Trans.* VIII. 138—140.

May 14, 1838.

By Joh. Tozer, B.A. (Gonv. and Cai.): On the application of mathematics to calculate the effects of the use of machinery on the wealth of a community. *Trans.* VI. 507—522.

By Duncan Farquharson Gregory, B.A. (Trin.): On the real nature of symbolical algebra.

By Professor Miller: On measures of spurious rainbows.

May 28, 1838.

By Professor Miller: An account of experiments illustrating the unequal expansion of crystals by heat.

By Ri. Tho. Lowe, M.A. (Chr.). On the Botany of Madeira[1]. *Trans.* VI. 523—551.

November 12, 1838.

By Professor Whewell: On certain rude kinds of architecture.

November 26, 1838.

By Duncan Farquharson Gregory, B.A. (Trin.): On the logarithms of negative quantities.

By Professor Henslow: On the formation of mineral veins, illustrated by a specimen.

December 10, 1838.

By Hamnett Holditch, M.A. (Gonv. and Cai.): On rolling curves (communicated by Professor Willis). *Trans.* VII. 61—86.

By Ri. Wellesley Rothman, M.A. (Trin.): On the climate of Italy.

By Ri. Tho. Lowe, M.A. (Chr.): An additional note on the Flora of Madeira.

By Professor Henslow: On the structure of wasps' nests, illustrated by specimens.

February 18, 1839.

By Ri. Wellesley Rothman, M.A. (Trin.): On the climate of Italy (concluded).

By Ri. Potter, B.A. (Qu.): On the determination of the value of (λ), the length of an undulation of light.

By Geo. Green, B.A. (Gonv. and Cai.): Appendix to a former paper on waves, read 15 May, 1837. *Trans.* VII. 87—95.

March 4, 1839.

By Will. Hopkins, M.A. (Pet.): On the geology of England and France in the neighbourhood of the Channel.

[1] This paper is not mentioned in the *Minutes*. The date assigned to it is derived from the *Transactions*.

March 18, 1839.

By Sam. Earnshaw, M.A. (Joh.): On the equilibrium of a system of particles. *Trans.* VII. 97—112.

By Geo. Biddell Airy, M.A. (Trin.): On the diurnal changes of the variation of the magnetic needle.

April 22, 1839.

By Duncan Farquharson Gregory, B.A. (Trin.): On photogenic drawings.

By Professor Sedgwick: On the geology of Cornwall and Devon.

May 6, 1839.

By Professor Miller: On the calculation of halos, according to Fraunhofer's theory.

By Geo. Green: Note on reflection and refraction. *Trans.* VII. 113—120.

By Duncan Farquharson Gregory, B.A. (Trin.): On chemical classification.

May 20, 1839.

By Geo. Green, B.A. (Gonv. and Cai.): On the motion of light through crystallised media. *Trans.* VII. 121—140.

By Dav. Tho. Ansted, B.A. (Jes.): On the tertiary formations of Switzerland. *Trans.* VII. 141—152.

By Professor Whewell: An account of observations made with his Anemometer since May, 1837.

November 11, 1839.

By Professor Whewell: On a new theory of the Tides. *Phil. Mag.* 1839, ii. 476.

November 25, 1839.

By Professor Sedgwick: On the geology of northern Germany, east and west of the Rhine.

December 9, 1839.

By Aug. De Morgan, B.A. (Trin.): On the foundations of Algebra. *Trans.* VII. 173—187.

By Dav. Tho. Ansted, B.A. (Jes.): On the geology of the Transition Rocks in the north-east of Bavaria and the Principality of Reuss.

By Will. Webster Fisher, M.B. (Down.): On the malformation of certain parts of the nervous system.

March 2, 1840.

By Geo. Biddell Airy, M.A. (Trin.): On a new construction of the Going Fusee, applied in the Northumberland telescope. *Trans.* VII. 217—277.

By Ch. Pritchard, M.A. (Joh.): On the achromatism of the telescope.

March 16, 1840.

By Aug. De Morgan, B.A. (Trin.): On the foundation of algebra.

By Joh. Tozer, M.A. (Gonv. and Cai.): On some doctrines of Political Economy. *Trans.* VII. 189—196.

March 30, 1840.

By Phil. Kelland, M.A. (Qu.): On the quantity of light intercepted by a grating placed before a lens. *Trans.* VII. 153—171.

May 4, 1840.

By Phil. Kelland, M.A. (Qu.): On the Law of molecular attraction.

May 18, 1840.

By Dav. Tho. Ansted, M.A. (Jes.): On the Green Sandstone formation of Blackdown, Devon.

By Professor Miller: On the structure of the Heliotropes of Gauss, Steinheil, and Schumacher.

June 1, 1840.

By Will. Hopkins, M.A. (Pet.): On certain geological phenomena of elevation, and their connection with the formation of volcanoes. *Phil. Mag.* 1840, ii. 154.

November 16, 1840.

By Roderick Impey Murchison: On the geology of Russia.

By Geo. Biddell Airy, M.A. (Trin.): On an optical fact, and its explanation on the undulatory theory.

November 30, 1840.

By Professor Henslow: On the diseases of wheat.

December 14, 1840.

By Aug. De Morgan, B.A. (Trin.): On the composition of forces.

By Professor Whewell: On the equilibrium of oblique arches.

February 22, 1841.

By Professor Whewell: Additional remarks on oblique arches.

By the same: Is all matter heavy? *Trans.* VII. 197—207.

March 8, 1841.

By Joh. Tozer, M.A. (Gonv. and Cai.): On some mathematical formulæ for determining the permanent effects of emigration and immigration on numbers. *Phil. Mag.* 1841, i. 318.

March 22, 1841.

By Professor Miller: On supernumerary rainbows. *Trans.* VII. 277—286.

April 26, 1841.

By Professor Challis: On the resistance of air to a pendulum with a spherical bob. *Trans.* VII. 333—353.

May 10, 1841.

By Professor Willis: On the arrangement of the joints of crustaceous animals.

By the same: On the original nomenclature of Gothic mouldings.

May 24, 1841.

By Professor Challis: On a new kind of interference of light.

November 15, 1841.

By Professor Sedgwick: An account of the comparative classification of the older strata in the British Isles.

November 29, 1841.

By Aug. De Morgan, M.A. (Trin.): On the foundation of algebra. *Trans.* VII. 287—300.

By Jos. Power, M.A. (Trin. Hall): On the late accident on the Brighton railway. *Trans.* VII. 301—317.

By Joh. Fre. Stanford, B.A. (Chr.): On a newly invented locomotive.

December 13, 1841.

By Will. Hopkins, M.A. (Pet.): On the forms of the isothermal surfaces within the earth; and on the thickness of the earth's solid crust, supposing the central portion to be fluid.

By Aug. De Morgan, B.A. (Trin.): On the foundation of algebra (continued).

February 14, 1842.

By Rob. Leslie Ellis, M.A. (Trin.): On the foundations of the doctrine of chances. *Trans.* VIII. 1—6.

February 28, 1842.

By Rob. Leslie Ellis, M.A. (Trin.): On the doctrine of chances (concluded). *Trans.* VIII. 1—6.

March 14, 1842.

By Mr Taplin: On the solution of a cubic equation.

By Professor Whewell (Master of Trinity College): Are cause and effect simultaneous or successive? *Trans.* VII. 319—331.

April 11, 1842.

By Professor Challis: On the differential equations of fluid motion. *Trans.* VII. 371—396.

By Professor Owen: On the fossil remains of a new genus of Saurians called *Rhynchosaurus*, discovered in the New Red Sandstone of Warwickshire. *Trans.* VII. 355—369.

April 25, 1842.

By Matth. O'Brien, M.A. (Gonv. and Cai.): On the propagation of luminous waves in the interior of transparent bodies. *Trans.* VII. 397—437.

By Geo. Gabriel Stokes, B.A. (Pemb.): On the steady motion of incompressible fluids. *Trans.* VII. 439—453.

May 9, 1842.

By Professor Kelland: On the motion of glaciers.

By the same: On the laws of fluid motion.

By Jos. Power, M.A. (Trin. Hall): On fluid motion. *Trans.* VII. 455—464.

By Professor Miller: An account of the *Dioptrische Untersuchungen* of Gauss.

November 14, 1842.

By Professor Fisher: On the development of the spinal ganglia in animals, and on the malformation of various portions of the nervous system in Man. *Phil. Mag.* 1842, ii. 485.

November 28, 1842.

By Matth. O'Brien, M.A. (Gonv. and Cai.): On the intensity of reflected and refracted light, the absorption of light, and the stability of the luminous æther. *Trans.* VIII. 7—26.

December 12, 1842.

By Will. Hopkins, M.A. (Pet.): On the glaciers of the Bernese Alps.

February 20, 1843.

By Art. Cayley, B.A. (Trin.): On some properties of determinants. *Trans.* VIII. 75—88.

By Matth. O'Brien, M.A. (Gonv. and Cai.): On the absorption of light by transparent media. *Trans.* VIII. 27—30.

March 6, 1843.

By Professor Challis: On a new general equation in Hydro-dynamics. *Trans.* VIII. 31—43.

By Geo. Kemp, M.B. (Pet.): On the nature of the biliary secretion: to shew that the bile is essentially composed of an electro-negative body, in chemical combination with one or more inorganic bases. *Trans.* VIII. 44—49.

March 20, 1843.

By Professor Sedgwick: On Professor Owen's memoir on the skeleton of the *Mylodon*; and on the structure and habits of certain extinct genera of gigantic Sloths.

May 1, 1843.

By Professor Challis: On the comet of 1843.

By Will. Williamson, M.A. (Cla.): Two letters on the same subject.

By Will. Hopkins, M.A. (Pet.): On the motion of glaciers. *Trans.* VIII. 50—74.

May 15, 1843.

By Hamnett Holditch, M.A. (Gonv. and Cai.): On small finite oscillations. *Trans.* VIII. 89—104.

By Professor Willis: On the vaults of the Middle Ages.

May 29, 1843.

By Geo. Kemp, M.B. (Pet.): On the relation between organic and organized bodies; with some remarks on the theory of organic combinations as proposed by Laurent.

By Geo. Gabriel Stokes, B.A. (Pemb.): On some cases of fluid motion. *Trans.* VIII. 105—137.

October 30, 1843.

By Will. Hopkins, M.A. (Pet.): An account of the large reflecting telescope which the Earl of Rosse is now constructing; with an account of the manner in which its 6 feet speculum has been prepared.

November 13, 1843.

By Professor Sedgwick: An account of the structure and relations of the slate rocks of North Wales.

Between 1831 and 1843 the Proceedings of the Society were reported—somewhat irregularly—in the *Philosophical Magazine*. The notices, as a general rule, are extremely brief; but I have thought it worth while to add references to those papers that are not printed in the Society's *Transactions*

whenever the abstracts give details. Moreover, this Journal preserves the titles, and brief abstracts, of four papers not noticed in the *Minutes* of the Society. These I have included between square brackets. They belong to the meetings held May 12, December 8, 1834; May 15, 1837.

P. S. Since writing the above sketch, I have discovered that a somewhat similar scheme for the establishment of a Philosophical Society had been projected in 1782. "The death of some persons interested in the plan, and several accidents, occasioned the scheme to be postponed till February 18th, 1784," when Professor Milner, Mr Farish, and some others "associated themselves under certain laws and regulations." They were presently joined by several of the most distinguished men in the University, among whom occurs the illustrious name of Porson, and a volume of "Tracts, Philosophical and Literary, by a society of gentlemen of the University of Cambridge" was projected, but never published, though two of the contributions were printed. "This little society of learned men, not being adequately supported, was dissolved about the close of the year 1786[1]." One promoter at least of this noble, though unsuccessful, attempt, Mr Farish, Jacksonian Professor from 1813 to 1837, became a member of the Philosophical Society.

[1] This information is derived from: *Memoirs of John Martyn, F.R.S., and of Thomas Martyn, B.D., F.R.S., F.L.S., Professors of Botany in the University of Cambridge.* By G. C. Gorham. 8vo. Lond. and Camb. 1830, p. 165.

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PROCEEDINGS

OF THE

Cambridge Philosophical Society.

October 28, 1889.

ANNUAL GENERAL MEETING.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

THE following Fellows were elected Officers and new Members of Council for the ensuing year:

President:

Mr. J. W. Clark.

Vice-Presidents:

Dr Routh, Prof. Babington, Prof. Liveing.

Treasurer:

Mr Glazebrook.

Secretaries:

Mr Larmor, Mr Harmer, Mr Forsyth.

New Members of Council:

Prof. Cayley, Prof. Darwin, Prof. Lewis, Dr Gaskell.

The names of the benefactors of the Society were recited by the Secretary.

On the motion of the PRESIDENT, seconded by the TREASURER, it was resolved:—That during the ensuing session the meeting of the Council of the Society be fixed for 4 o'clock in the afternoon; that the meeting of the Society be fixed for 4.30, when tea will be provided, and that the formal business of the meeting be taken at 5 o'clock.

The President delivered the following address:

I rise, according to custom, to say a few words at what would, under ordinary circumstances, be the close of my first year of office as your President. It happened, however, that in consequence of the lamented death of Mr Coutts Trotter I was elected on Monday, 30 January, 1888, instead of at the usual time in November. I have therefore had the honour of holding the office for a year and nine months.

During that period the Society has pursued the even tenour of its way, undisturbed by revolutions or dissensions, but at the same time giving signs of vigorous and healthy life. It is evident that we can no longer expect that a majority of those elected to College Fellowships in this University will seek the further distinction of our Fellowship, almost as a matter of course, as they used to do; but it may still be a subject of congratulation that our annual recruits make up in quality for defects in quantity; and, if I am not too sanguine, I think that the ancient popularity of our Society may, at any rate to some extent, be revived.

Our meetings have been well attended; but it seems to me that it might be possible to render them still more attractive, and so to make them fortnightly gatherings of those who, being interested in scientific pursuits, are anxious to meet persons of tastes similar to their own. With this object in view a proposal will be submitted to you for changing the hour of meeting. You remember that in 1881–82 the hour was changed from eight in the evening to three in the afternoon. That change was received with approbation; but since then it has been represented to the Council that a further change to a somewhat later hour in the afternoon would now be more convenient. It is now, therefore, proposed to meet at half-past four; and further, to offer, before the meeting, that refreshment which, in all circles, whether rich or poor—scientific, literary, or social—offers a resting-place between lunch and dinner—or between dinner and supper—namely, tea. It was remarked long ago that "great events from little causes spring;" so let us hope that this innocent beverage, which has not, as yet, fallen under the ban of any school of reformers,

may operate as a charm to bring our members together, and promote the ends we have in view.

There is another matter to which I would briefly allude—our Library. The importance of having a central scientific Library in these buildings, in addition to departmental libraries, has long been recognised in principle; and I cannot but think that when the practical usefulness of it becomes more widely known, members of the University will gladly support it by becoming Fellows of the Society. At present the number of students who use it steadily increases; but I am sorry to say that the power of the Society to support it is diminishing. In this year it became necessary to ask the Museums and Lecture Rooms Syndicate to defray the cost of certain periodicals which had heretofore been borne by the Society. This the Syndicate agreed to do; but I would remark that, while they fully recognise the value of the Library, and are willing to spend money liberally upon it, the funds at their disposal are by no means large, having regard to what they have to do with them. Meanwhile the Library is deficient in numerous works which are being continually asked for, and which neither the Society nor the Syndicate are rich enough to purchase.

The smallness of our funds operates to our disadvantage in another way—we are unable to illustrate our publications as fully as is desirable. You will, I am sure, give credit to those in charge of our *Proceedings* for the pains they bestow upon them, and for the praiseworthy rapidity with which they are circulated. A number of the *Proceedings* completing Vol. VI. is now on the table; and a part of the *Transactions*, completing Vol. XIV. is now ready.

I beg to return my most cordial thanks to the Council and the officers of the Society for the assistance they have rendered to me personally; and I am sure that you will recognise the admirable zeal with which they have discharged their duties to the Society. The Treasurer, Mr Glazebrook, should be specially congratulated on the success which has attended his efforts to obtain payment of arrears of subscriptions due to the Society.

Lastly, it is my duty to record the names of those Fellows of the Society whom we have lost by death during the past year. I will take them in the order of seniority.

The Rev. Richard Okes, D.D., Provost of King's College.

The Rev. Benjamin Hall Kennedy, D.D., Regius Professor of Greek.

William Henry Drosier, M.D., Fellow of Gonville and Caius College.

The Rev. Churchill Babington, D.D., formerly Fellow of St John's College.

John Reynolds Vaizey, M.A., Peterhouse.

Besides these, we have lost one honorary member:

James Prescott Joule, F.R.S.

It had been my intention to attempt a short biography of each of these; but I found that I had neither time nor materials to perform such a task efficiently. Moreover, they are for the most part too well known, and too intimately connected with this University, to need such commendation. It would, however, be a personal satisfaction to myself to remind you that in Mr Vaizey—who died from the results of an accident at the beginning of this year—the Biological School has lost an energetic worker, whose usefulness as a teacher had been already recognised, and who, had he lived, would probably have risen to eminence in his own special science, Botany.

The following Communications were made to the Society:

(1) *On Newton's description of orbits.* By CHARLES TAYLOR, D.D., Master of St John's College.

The Master drew attention to the fact that the problem of constructing a Conic Section to satisfy given conditions has been treated incidentally with great power and considerable completeness in the *Principia*. A comparison was made between the methods of Newton and the more modern methods: and some improvements were suggested. The way in which Newton passes from cases of real intersection of lines with conics to cases in which real points of intersection do not exist, strongly suggests the question whether he had possession of the idea of imaginary points, which is usually ascribed to a much later period.

(2) *On impulsive stress in shafting, and on repeated loading.* By Prof. KARL PEARSON, University College, London.

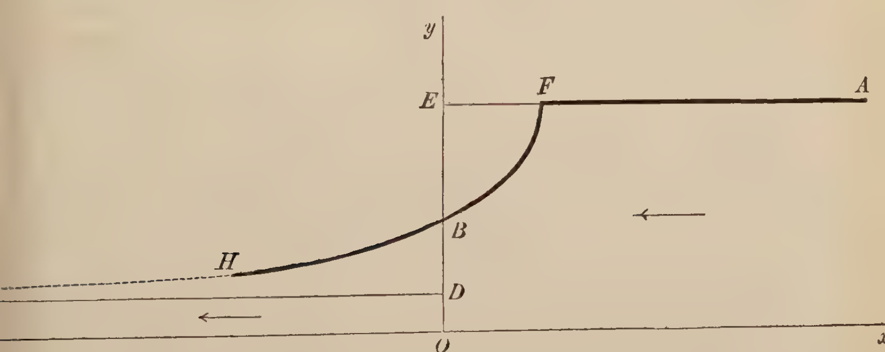
(3) *On Liquid Jets and the Vena Contracta.* By H. J. SHARPE, M.A., St John's College.

1. When liquid flows out of a vessel through an orifice, a liquid particle in contact with the vessel describes an ordinary stream-line as long as the particle is within the vessel, but the moment it escapes through the orifice, this stream-line suddenly becomes *also* a line of constant velocity, if no force acts on the liquid. In the solutions presently to be given, which are capable of infinite variety, the coincidence between the outer stream-line of the jet and a line of constant velocity is not (as in Kirchhoff's solutions) mathematically perfect, but (even near the orifice, where it is most imperfect) can be made, as will be seen from examples, *very close*, and as we pass along the jet, becomes very

rapidly nearly perfect. If the position of the orifice (subject however to the limitation that the coefficient of contractoin must be $>\frac{1}{2}$) be arbitrarily chosen, it will be seen that it can be so chosen as to make the coincidence as close as may be desired. If however, which seems the more correct course, it be chosen from a consideration presently to be given (Art. 5) it is not easy, I think, to say for certain whether the above-named coincidence can be made as close as may be desired. It is an interesting question which remains for solution. The motion is in two dimensions and everywhere finite. The species of vessel for which solutions are obtained may be described generally as a canal, whose sides up to a certain point are straight and then turn off *abruptly* at right angles into a curve towards an orifice, at the axis of the canal, on each side of which the fluid motions are symmetrical*. The jet ultimately approaches an asymptote parallel to its axis. The ratio of the breadth of the vessel to the ultimate breadth of the jet can be made anything we like, but we shall always suppose it an even integer.

2. Liquid is supposed to be flowing from right to left, roughly speaking parallel to the axis of x, which is taken as the axis of the vessel and jet. The stream-line $AFBH$ is taken for the boundary of the vessel. Fig. 1 may be taken as the type of the

FIG. 1.

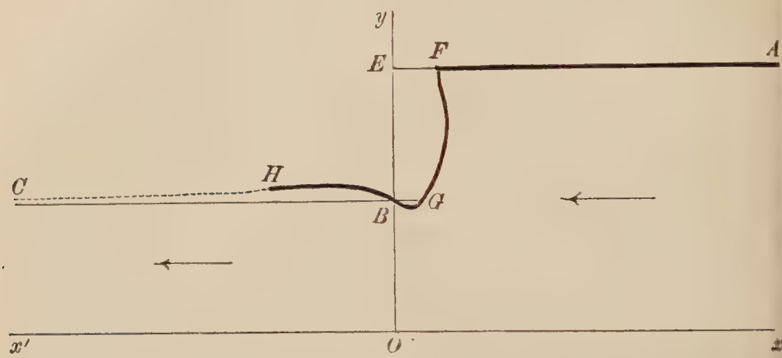


general case. OE is *always* π, and OB, OD submultiples of π. In figures of which fig. 2 is a type B and D coincide. At present

* It will be seen, however (Art. 6), by an obvious extension of the method, that it is not confined to canals having this peculiarity, but is applicable also to canals having flowing curvilinear boundaries.

the only limitation we shall put upon the position of the orifice H is that its ordinate must not exceed twice OD.

Fig. 2.



3. *Different* analytical expressions, containing an arbitrary number of arbitrary constants, will be assumed for the velocities on either side of Oy, but it will be shewn that they can be so chosen as to make the velocities on either side of Oy *continuous*, *and* leave any required number of arbitrary constants to satisfy conditions now to be given. It will be shewn that in all cases the equation to BHC can be expressed in the form (putting, for brevity, z for ϵ^x),

$$y = \alpha - c_1 z \sin y - \tfrac{1}{2} c_2 z^2 \sin 2y - \&c. \ldots\ldots\ldots\ldots (1),$$

and the velocities on the *left* of Oy in the form

$$\left.\begin{aligned} -\frac{u}{A} &= 1 + c_1 z \cos y + c_2 z^2 \cos 2y + \&c., \\ \frac{v}{A} &= c_1 z \sin y + c_2 z^2 \sin 2y + \&c., \end{aligned}\right\} \ldots\ldots\ldots (2),$$

where of course α is the ultimate value of Cx'. We are concerned only with the velocities at all points along HC, so that x and y in (1) are the same as x and y in (2). As z is less than 1 we can solve (1) so as to express y in a series of ascending powers of z. We shall have therefore at every point of HC to the second order of approximation

$$y = \alpha - c_1 z \sin \alpha + \&c. \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$\frac{1}{A^2}(u^2 + v^2) = 1 + 2c_1 z \cos \alpha + \&c. \ \ldots\ldots\ldots\ldots (4).$$

If $\alpha = \pi/2$ the coefficient of z vanishes, and it will be shewn that the remaining disposable constants can be so chosen as to make the coefficients of z^2, z^3, &c. (any desired number of them) also vanish.

Next suppose that $c_1 = 0$, then to the third order of approximation

$$\frac{1}{A^2}(u^2+v^2) = 1 + 2c_2 z^2 \cos 2\alpha + \&c.$$

If $\alpha = \pi/4$ the coefficient of z^2 vanishes, and if there are constants enough, the coefficients of z^3, z^4, &c. could be made to vanish. And generally if $c_1, c_2 \ldots c_{m-1}$ vanish, then to the $(m+1)$th order of approximation

$$\frac{1}{A^2}(u^2+v^2) = 1 + 2c_m z^m \cos m\alpha \quad\ldots\ldots\ldots\ldots(5),$$

and if $\alpha = \pi/2m$ the coefficient of z^m vanishes, and the coefficients of z^{m+1} &c. could be made to vanish.

We have supposed at first, for simplicity, *all* the powers of z in (1) to exist, but in every case we shall find that many are wanting. For instance, after the first few terms, only *even* multiples of y may occur—or again multiples of $6y$. Again, even if (1) is complete, yet it will be found that if $\alpha = \pi/2$, (3) will consist only of odd, and (4) of even powers of z. This consideration not only simplifies the work, but also enables us with greater ease to make (u^2+v^2) constant to a higher degree of approximation.

4. We shall now shew how to make the velocities on either side of Oy continuous. We shall take the case to which fig. 2 refers, where OB is $\pi/2$ and where the ultimate value of Cx' is OB. I give this case, not only because it is one of the first I solved, but because I believe it exhibits the method under the greatest disadvantages, and yet it will be found that the approximation obtained is very close.

We will take for the stream function ψ, and the velocities on the *left* of Oy,

$$\left.\begin{aligned} -\frac{d\psi}{dy} &\equiv -u = a_1\epsilon^{x}\cos y + a_3\epsilon^{3x}\cos 3y + \Sigma a_{2n}\epsilon^{2nx}\cos 2ny + A \\ -\frac{d\psi}{dx} &\equiv v = a_1\epsilon^{x}\sin y + a_3\epsilon^{3x}\sin 3y + \Sigma a_{2n}\epsilon^{2nx}\sin 2ny \end{aligned}\right\}\ldots(6),$$

Σ indicates summation for all integral powers of n from 1 to ∞. Only *two odd* multiples of y appear in the above, but it must be distinctly understood (in fact it is the characteristic feature of the present method) that *any* number of *any* odd multiples of y could

be similarly used, and each assumption would give a distinct case. The same also must be understood for the stream function and velocities presently to be assumed on the right of Oy.

Then the equation to BHC is

$$a_1\epsilon^x \sin y + \tfrac{1}{3}a_3\epsilon^{3x}\sin 3y + \Sigma\frac{a_{2n}}{2n}\epsilon^{2nx}\sin 2ny$$
$$+ Ay = a_1 - \frac{a_3}{3} + \frac{A\pi}{2} \dots (7).$$

Since the ultimate breadth of the jet is supposed to be $\pi/2$, we must have

$$a_1 - \tfrac{1}{3}a_3 = 0 \dots\dots\dots (8).$$

It will be convenient to replace a_1, a_3 each by two new quantities, such that

$$a_1 \equiv \alpha_1 + A_1, \quad a_3 \equiv \alpha_3 + A_3 \dots\dots\dots (9).$$

Then when $x = 0$, we have at every point of OB, on the *left* of Oy

$$\left.\begin{aligned} -u &= (\alpha_1 + A_1)\cos y + (\alpha_3 + A_3)\cos 3y + \Sigma a_{2n}\cos 2ny + A \\ v &= (\alpha_1 + A_1)\sin y + (\alpha_3 + A_3)\sin 3y + \Sigma a_{2n}\sin 2ny \end{aligned}\right\} (10).$$

On the *right* of Oy assume for the velocities

$$\left.\begin{aligned} -\frac{d\chi}{dy} &\equiv -u = b_1\epsilon^{-x}\cos y + b_3\epsilon^{-3x}\cos 3y + \Sigma b_{2n}\epsilon^{-2nx}\cos 2ny + B \\ -\frac{d\chi}{dx} &\equiv v = -b_1\epsilon^{-x}\sin y - b_3\epsilon^{-3x}\sin 3y - \Sigma b_{2n}\epsilon^{-2nx}\sin 2ny \end{aligned}\right\} (11).$$

It will be convenient to replace b_1, b_3 each by two new quantities, such that

$$b_1 \equiv \alpha_1 - A_1, \quad b_3 \equiv \alpha_3 - A_3 \dots\dots\dots (12).$$

Then when $x = 0$, we have at every point of OB, on the *right* of Oy,

$$\left.\begin{aligned} -u &= (\alpha_1 - A_1)\cos y + (\alpha_3 - A_3)\cos 3y + \Sigma b_{2n}\cos 2ny + B \\ v &= -(\alpha_1 - A_1)\sin y - (\alpha_3 - A_3)\sin 3y - \Sigma b_{2n}\sin 2ny \end{aligned}\right\} (13).$$

By Fourier's theorem, suppose we have, from $y = 0$ to $\pi/2$,

$$2A_1\cos y + 2A_3\cos 3y = p + \Sigma p_{2n}\cos 2ny \dots\dots\dots (14).$$

This will be true at both limits. Then the first equation of (10) will be identical with the first equation of (13) if

$$a_{2n} - b_{2n} + p_{2n} = 0 \dots\dots\dots (15),$$

$$A - B + p = 0 \dots\dots\dots (16).$$

Again, by Fourier's theorem, suppose we have from $y=0$ to $\pi/2$

$$2\alpha_1 \sin y + 2\alpha_3 \sin 3y = \Sigma q_{2n} \sin 2ny \text{.............(17).}$$

This will be true at both limits if

$$\alpha_1 - \alpha_3 = 0 \text{..........................(18).}$$

Then the second equation of (10) will be identical with the second equation of (13) if

$$a_{2n} + b_{2n} + q_{2n} = 0 \text{ ....(19).}$$

Then if the constants satisfy equations (15), (16), (18), (19) the motions on the left and right of OB will be continuous.

From (11) the equation to AFB is

$$b_1 \epsilon^{-x} \sin y + \tfrac{1}{3} b_3 \epsilon^{-3x} \sin 3y + \Sigma \frac{b_{2n}}{2n} \epsilon^{-2nx} \sin 2ny$$

$$+ By = b_1 - \frac{b_3}{3} + \frac{B\pi}{2} \text{...(20).}$$

5. We will now suppose that

$$b_1 - \frac{b_3}{3} = \frac{B\pi}{2} \text{ .... (21).}$$

When (21) is fulfilled, the stream-line AFB will consist of an infinite straight line AF, whose ordinate is π, and a curved portion FB. The peculiarity at F will be presently explained. It must be carefully observed that AF is not an asymptote.

It will be found that we get the following relations

$$a_1 = \alpha_1 + A_1, \qquad a_2 = -\frac{8}{15\pi}(17\alpha_1 + 16A_1),$$

$$a_3 = 3\alpha_1 + 3A_1, \qquad a_4 = -\frac{8}{105\pi}(61\alpha_1 + 64A_1),$$

$$\left.\begin{matrix} a_{2n} \\ b_{2n} \end{matrix}\right\} = \frac{2}{\pi} \cos n\pi \, . \left\{ \alpha_1 \left(\frac{4n}{4n^2-1} - \frac{4n \pm 12}{4n^2-9} \right) + A_1 \left(\frac{\pm 2}{4n^2-1} \mp \frac{18}{4n^2-9} \right) \right\},$$

$$b_1 = \alpha_1 - A_1, \qquad b_2 = \frac{8}{15\pi}(\alpha_1 + 16A_1),$$

$$b_3 = -\alpha_1 - 3A_1, \qquad b_4 = \frac{8}{105\pi}(29\alpha_1 + 64A_1),$$

$$A = \frac{16\alpha_1}{3\pi}, \quad B = \frac{8\alpha_1}{3\pi} \text{ ....(22).}$$

It will be noticed that for moderately large values of n, a_{2n} and b_{2n} ultimately vary as $1/n^2$, so that all the series employed

are convergent even for points very near Oy. I believe this property is common to all the solutions obtained by the present method. It is interesting to notice that unless we assumed the equation (18) to hold, we could not establish the obvious relation $A = 2B$.

6. We will now make the curve HC as far as possible identical with a line of constant velocity. It will be convenient to put for shortness $\epsilon^x \equiv z$,

$$a_1/A \equiv c_1, \quad a_2/A \equiv c_2, \quad a_3/A \equiv c_3, \text{ \&c.}$$

Then (7) can be written

$$y = \frac{\pi}{2} - c_1 z \sin y - \tfrac{1}{2} c_2 z^2 \sin 2y - \text{\&c.} \ldots\ldots (23),$$

and from (6),

$$\left.\begin{aligned} -\frac{u}{A} &= 1 + c_1 z \cos y + c_2 z^2 \cos 2y + \text{\&c.} \\ \frac{v}{A} &= c_1 z \sin y + c_2 z^2 \sin 2y + \text{\&c.} \end{aligned}\right\} \ldots\ldots\ldots\ldots (24).$$

In (23) expressing y in terms of z, we have

$$y = \frac{\pi}{2} - c_1 z + (\tfrac{1}{2} c_1^3 - c_1 c_2 + \tfrac{1}{3} c_3) z^3 + \text{\&c.} \ldots\ldots (25).$$

It will be found that this series consists only of *odd* powers of z, the simplification arising from our having taken $\pi/2$ for the ultimate breadth of the jet. If we now substitute the above value of y in (24) and form the value of $u^2 + v^2$, we shall get

$$\frac{1}{A^2}(u^2 + v^2) = 1 + (3c_1^2 - 2c_2) z^2$$
$$+ (-\tfrac{4}{3} c_1^4 + 8c_1^2 c_2 + c_2^2 - \tfrac{26}{3} c_1 c_3 + 2c_4) z^4 + \text{\&c.} \ldots (26),$$

the series consisting only of *even* powers of z.

If we want to carry the approximation to the fourth order, we shall have to cause the coefficient of z^2 here to vanish. This will give us for determining the ratio of A_1 to α_1 the equation

$$(\alpha_1 + A)^2 + \frac{256\alpha_1}{135\pi^2}(17\alpha_1 + 16A_1) = 0.$$

Solving this equation we get

$$A_1/\alpha_1 = -1{\cdot}0635 \text{ or } -4{\cdot}0127 \text{ nearly} \ldots\ldots\ldots (27).$$

Either of these gives a solution. If we take the first, we see from the equations (22) at the end of Art. 4, that all the quantities a_1, a_2, a_3, a_4 are small, therefore the curve BHC clings all along very closely to its asymptote.

From equations (6) or (11) we shall get for the direction of the fluid motion at B,

$$\frac{v}{u}=\frac{3\pi}{10}\left(1+\frac{A_1}{a_1}\right).$$

For the first solution this becomes $-\cdot 06$, and for the second $-2\cdot 84$.

If we wished to carry the approximation to the sixth order, we should have to introduce terms with the sine and cosine of $5y$ in (6) and (11) and make the coefficient of z^4 in (26) vanish. This solution would involve *two* arbitrary constants.

By making *in addition* the coefficient of z^6 in (26) vanish we could carry the approximation to the 8th order, and the solution would involve *one* arbitrary constant.

7. We proceed now to discuss the solutions obtained. And first for the point F, where there is a peculiarity which was pointed out to me by Sir George Stokes.

From (20) the equation to AFB can be written (putting for shortness z for ϵ^{-x}),

$$B(y-\pi)=b_1 z\sin(y-\pi)+\tfrac{1}{3}b_3 z^3\sin 3(y-\pi)$$
$$-\Sigma\frac{b_{2n}z^{2n}}{2n}\sin 2n(y-\pi)\ldots\ldots(28).$$

Dividing out by $(y-\pi)$ and then putting $y=\pi$, we get for determining the abscissa of F,

$$B=b_1 z+b_3 z^3-\Sigma b_{2n}z^{2n}\ldots\ldots\ldots\ldots\ldots\ldots(29).$$

For the first solution in (27) this gives us $z=\cdot 58$, from which we get $x=23/43$ nearly.

Further it will be seen from (28) that *for points near F, z* does not vary, for on account of the values of z and b_{2n} (see end of Art. 4) the series converges pretty rapidly. Therefore the stream-line turns sharply at right angles at F. This curious result may be further corroborated simply by comparing the first equation of (11) and (29) from which we at once get $u=0$ at F. It may be noticed that *we are not compelled* to make the ultimate ordinate of the stream-line on the right of Oy equal to π. For instance, in the present case, we arbitrarily assumed (21) to hold. If we did *not* assume this, the outer stream-line on the right of Oy would have no sharp corner. The ultimate ordinate however, whatever it is, *must not exceed* π, otherwise the motion would be discontinuous.

8. Kirchhoff has shewn (Lamb's *Motion of Fluids*, Art. 96) that at such a point as H (where the motion of the fluid having

been restricted suddenly becomes free) the radius of curvature should be zero in the true solution. In the present approximate solutions we cannot of course expect this to hold. But I am inclined to believe that in *all* solutions obtained by the present method there will be found on the line BHC (near where we might expect the orifice to be) a point where *the radius of curvature is a minimum*, and here I propose to place the point H. At any rate there is such a point in the case of the first solution in (27), as we proceed to shew.

From (25) putting in it $c_2 = \frac{3}{2}c_1^2$ and from (8) $c_3 = 3a_1$, the equation to BHC is

$$y = \frac{\pi}{2} - c_1 z + (-c_1^3 + c_1) z^3 + \&c. \qquad (30).$$

From (27), &c.

$$c_1 = \frac{a_1}{A} = \frac{a_1 + A_1}{A} = -\frac{3\pi}{16} \times \cdot 0635.$$

It is evident therefore that c_1 is small and that (30) may be approximately written

$$y = \frac{\pi}{2} - c_1 (z - z^3) + \&c.$$

As dy/dx is always small all along BHC, we may get sufficiently near the point required by finding the point where d^2y/dx^2 is a maximum. We see at once that this is got from the equation z^2 is equal to $1/27$ which gives $x = -71/43$ nearly for the abscissa of H. We can readily see that at H the curve is *convex* to the axis and that a *maximum* value of d^2y/dx^2 has been obtained, also that the coefficient of contraction is ·99545.

9. It will now be interesting to calculate the limits of error in the velocity at the point H. From (26) it can be shewn that the coefficient of z^4 reduces to

$$\tfrac{155}{12} c_1^4 - 26 c_1^2 + 2c_4.$$

This is equal to ·1654 nearly. Therefore at the point H only ·000011 of the velocity is variable. Of course as we pass to the left of H this small proportion rapidly diminishes.

10. If we examine equation (20) we shall find that the curve FGB cuts the line $y = \pi/2$ in a point G whose abscissa is about $3/43$. Also if we imagine two points on the same curve whose ordinates are $4\pi/6$ and $5\pi/6$ we shall find that their respective abscissae are about $16/43$ and $36/43$.

November 11, 1889.

DR GASKELL IN THE CHAIR.

The following Communications were made:

(1) *On the Varieties and Geographical Distribution of the common Dog-Whelk* (*Purpura lapillus* L.). By A. H. COOKE, M.A., King's College.

THE author, while unable to advance any satisfactory theory to account for *colour* variation, held that variations of *form* largely depended upon the station occupied by the animal. Shells occurring in exposed situations (e.g. Land's End, Scilly Islands, coasts of N. Devon and Cornwall) were stunted, with a short spire and large mouth, the latter being developed in order to increase the power of adherence to the rock, and of resistance to wave force. Shells occurring in sheltered situations, estuaries, narrow straits, &c. where there was no severe wave force to encounter, were of great size, spire well developed, mouth small in proportion to area of shell. This view was illustrated by series of specimens collected at various points on the British coasts.

With regard to the question of *geographical distribution* it was shewn that *Purpura lapillus* (a "north temperate" species) occurred on the East Asiatic coasts from Behring's Straits to Hakodadi (41°), on West European coasts from North Cape to Mogador (32°), not entering the Mediterranean, on East American coasts from Greenland to Newhaven (42°), and on West American coasts (assuming the identity of the West American Purpuras with *lapillus*) from Alaska to Margarita Bay (24°). Thus on the two western coasts it had a far more southern range than on the two eastern. The author regarded this fact as due to the direct influence of the surface temperature of the ocean. The mean annual temperature (taken from the Meteorological Society's charts) of the surface water at Hakodadi was 52°, with an extreme range of 25°; that of Mogador was 66°, extreme range only 8°; that of Newhaven was 52°, extreme range 30°; that of Margarita Bay 73°, extreme range only 5°. Violent changes of temperature were fatal to life, zones where such changes occurred acted as barriers to distribution; it was possible on the other hand for an organism to bear a *gradual* change from cold to extreme heat. On the western coasts of Europe and America the change from cold to heat was very gradual, hence the Purpura had been able to creep as far south as 32° in the one case and 24° in the other; while on the opposite eastern coasts, where the Atlantic and Pacific Gulf-streams caused a sudden change in the temperature of the surface-water, the species was barred back at a point many degrees further north.

(2) *On the increase in thickness of the stem of the Cucurbitaceæ.* By M. C. POTTER, M.A., St Peter's College.

THE Order *Cucurbitaceæ* consists for the most part of herbaceous plants climbing by means of tendrils; like many other climbing plants, the members of this order have an anomalous distribution of the fibro-vascular bundles in the stem; the bundles being arranged in two concentric rings, and each individual bundle being bicollateral, with phloem both on its external and internal sides. (Fig. 1.)

The structure of these stems was first described by Hartig* and then by von Mohl† and has been the subject of investigation by various botanists, most of whom have confined their attention to the structure and contents of the sieve tubes. Bertrand‡ however has described the manner in which a cambium between the xylem and both inner and outer phloem adds respectively both xylem and phloem to the bundle, whilst Petersen§, in his article on Bicollateral Bundles, gives a short account of the increase of the stem of *Zehneria suavis;* stating that there is no interfascicular cambium, but that while the bundles increase the cells of the medullary rays increase passively in a radial direction and finally divide. With this statement de Bary‖ agrees. Fischer¶ also says that there is no interfascicular cambium present whereby these stems can increase in thickness. The fact that all the investigated species of *Cucurbitaceæ* have been herbaceous explains why the interfascicular cambium has hitherto not been described.

Lately I have had an opportunity of investigating the woody stems of *Cephalandra indica* (Naud.), *Trichosanthes villosa* (Bl.) and *T. anamalayana* (Bedd.), and find that they increase by a well-marked interfascicular cambium. The stems of these plants agree with those of the other members of the order in possessing the two rings of bicollateral bundles (fig. 1), but differ in having no ring of sclerenchymatous tissue between the epidermis and vascular bundles. The structure of these species being similar in all respects it will only be necessary to describe the stem of *Cephalandra indica.*

Cephalandra indica, a climbing woody plant, reaches to a considerable height and its stem, which is perennial, attains to several inches in diameter. In its young stage the stem contains

* *Bot. Zeit.* 1854.
† *Bot. Zeit.* 1855.
‡ "Theorie du faisceau," *Bull. Sci. du département du Nord*, 1880.
§ *Engler Bot. Jahrb.* VIII. p. 374.
‖ *Comparative Anatomy*, English edition, p. 456.
¶ *Untersuchungen über das Siebröhren-system der Cucurbitaceen*, p. 6.

ten fibro-vascular bundles in two concentric rings, the five inner bundles being larger than the five outer ones (fig. 1). The stem increases slowly for some time, because the cambium on the outside as well as on the inside of each bundle adds xylem and phloem to the bundle; the cells of the medullary rays increase radially and divide. Thus far *Cephalandra* is similar to the herbaceous members of the order. The latter, being annuals or at least not requiring a continual enlargement of the stem, have no provision for such increase, whilst *Cephalandra* being woody and perennial requires means whereby its stem can continue to grow, and therefore by division of the cells of the medullary rays adjacent to the bundles and contiguous to the outer cambium cells there is formed an interfascicular cambium (fig. 2*a*). This interfascicular cambium soon stretches from bundle to bundle across the medullary rays (see dotted line fig. 1), and its development agrees with that of normal Dicotyledons. After the completion of the ring of interfascicular cambium, xylem is formed on the exterior of the existing xylem, and phloem on the interior of the outer phloem and secondary medullary rays are formed in the xylem in the normal manner (Plate II. fig. 4 mr_2). The stem therefore grows in the same manner as a normal Dicotyledon. The cambium which is placed on the inner side of the xylem ceases to produce new elements about the time the interfascicular cambium is formed, and finally disappears; so that the internal phloems are left at the centre of the stem and do not undergo further increase.

The above description shews that the mode of increase of the stem of *Cephalandra indica* corresponds exactly with that of a normal Dicotyledon in which no intermediate bundles are formed, inasmuch as each possesses a cambium which forms on its inside xylem and medullary rays, and on its outside phloem and parenchyma. But the stem of the former differs in the primary arrangement of the bundles in two rings and in having some phloem on the central side of the xylem. It would seem therefore that de Bary's* view that the two concentric rings of bundles in the stem behave as a single ring curving alternately outwards and inwards is correct.

EXPLANATION OF PLATES I AND II.

Fig. 1. Diagram shewing position and relative size of the fibro-vascular-bundles of the stem of *Cephalandra indica*. *Ph* = Phloem, *Xy* = Xylem, the wavy line indicating where the interfascicular cambium will be formed.

* *Comparative Anatomy*, English edition, p. 456.

Fig. 2. Portion of stem of *Cephalandra indica* shewing two interior and one exterior fibro-vascular-bundle. *a* cells dividing to form the cambium. *C* centre of stem. *Si* sieve-tubes. *P* the periphery.

Fig. 3. Portion of an older stem of *Cephalandra indica*. *c* centre of stem. *Si* sieve-tubes, *cam* cambium, *ph* position of interior phloem of two other fibro-vascular-bundles, mr_1 primary medullary ray, mr_2 secondary medullary ray.

(3) *On the Spinning Apparatus of Geometric Spiders.* By C. Warburton, B.A., Christ's College.

The structure of the external and internal spinning organs of the *Epeiridæ* was described, and the special functions of the several distinct kinds of spinning glands investigated.

The Ampullaceal glands were shewn to be the sources of the framework and radial lines of the geometric web. The Acinate and Piriform glands are those mainly used in binding up captured insects.

A Spider's line is not composed of many strands interwoven or coalescent, as has been hitherto believed. It usually consists of two or four non-adherent threads, and when more are present they do not fuse, but remain distinct, although contiguous.

The foundation line of the spiral consists of two strands only, not adhering on account of their own viscidity, but enveloped in a common viscid sheath which subsequently breaks up into bead-like globules, and which is probably furnished by the aggregate glands.

(4) *A new result of the injection of ferments.* By E. H. Hankin, B.A., St John's College (Junior George Henry Lewes Student).

The following experiments were performed in Professor Koch's laboratory at the Hygienisches Institut, Berlin. Although the theoretical considerations that led me to perform these experiments are by no means proved by them, the results appear to be sufficiently interesting to publish in detail.

Experiment 1. Three rabbits, Nos. 11, 12, and 13, were inoculated with virulent anthrax*. No. 13 served as control and died in 36 hours of typical anthrax. Nineteen hours after inoculation rabbits 11 and 12 were subjected to a further treatment. No. 11 had two cubic centimetres of a ·1 per cent. solution of trypsin injected into the lateral vein of its ear, and No. 12 had two cc. of a one per cent. solution of pepsin similarly injected†.

* These rabbits were of medium size, No. 11 weighed 855 grammes.

† The pepsin and trypsin employed were both obtained from Schering's Grüne Apotheke, Chaussée Strass, Berlin. In my later experiments I employed some very pure pepsin which I owe to the great kindness of Dr Theodor Weyl.

Rabbit 12 died 66 hours after its inoculation with anthrax. The anthrax bacilli in the lung, spleen, and lymph gland near seat of inoculation, instead of appearing in the form of short rods characteristic of virulent anthrax, were arranged for the most part in long chains as is usual with attenuated virus. The chains consisted of as many as 12 and sometimes even more joints. Two mice were inoculated from the heart-blood of this rabbit, and died of anthrax after the rather unusually long period of 60 hours. No. 11 died only 13 days after its inoculation. It had slight diarrhœa for some days before its death. Its spleen contained but few bacilli arranged in chains generally of six or seven but sometimes of as many as fifteen joints.

Three days after the inoculation of these rabbits another rabbit, 22, was inoculated from the same culture. The next day three cc. of one per cent. pepsin was injected intravenously. The rabbit died six days afterwards and its spleen shewed very few bacilli all arranged in chains.

Experiment 2. Three rabbits were inoculated with virulent anthrax. Two days afterwards only one was still alive. It was treated with four cc. of one per cent. trypsin injected into its ear vein. It died four days after its anthrax inoculation. The spleen contained numerous bacilli, which were for the most part arranged in chains, one of which contained as many as 24 joints. These chains shewed signs of degeneration, in that they stained very irregularly. In the same chain some joints were colourless while others were deeply tinted. The spleen of this rabbit was particularly large, and a great many of the cells contained more than one nucleus, while other nuclei had a dotted aspect. Apparently these appearances indicate an increased rate of nuclear division*.

Experiment 3. Professor Koch kindly inoculated for me five rabbits, Nos. 26 to 30, each with a large quantity of anthrax spores suspended in a normal salt solution. No. 29 served as control and died in 36 hours. I injected three to four cc. of one per cent. trypsin into the ear vein of Nos. 26, 27 and 28 directly afterwards. No. 26 died in the same time as the control. None of the bacilli were in very long chains, none of more than six joints were seen, while most of the bacilli were exceptionally short. This fact and also the way in which they were arranged on the slide suggested to me that the bacilli had at first grown in longer chains, but that at a later period (perhaps when the effect of the trypsin had passed away) they had broken up into separate segments.

* My observations were all made on fresh preparations of the spleen pulp, to which some dilute aqueous solution of methyl blue was generally added.

No. 27 died in about 50 hours. The bacilli in the spleen were rather longer than usual.

No. 28 died in $21\frac{1}{2}$ hours. Unfortunately I have mislaid my notes of its post mortem appearances.

No. 30 had four cc. of one per cent. trypsin injected intravenously the day after it was inoculated with anthrax. It died in 36 hours and the spleen shewed the bacilli arranged in long chains.

Experiment 4. Five rabbits, Nos. 34 to 38, were as before inoculated by Professor Koch with anthrax spores. Directly afterwards I administered intravenously from one to $3\frac{1}{2}$ cc. of ·05 per cent. solution of trypsin.

No. 38 was control and died between 50 and 60 hours after inoculation.

No. 34 weighed 1500 grammes and had one cc. of the above solution of trypsin. It died 51 hours after inoculation. Some of the bacilli were isolated, but many were in chains of more than 12 members.

No. 35 weighed 1507 grammes and had $3\frac{1}{2}$ cc. of the ·05 per cent. trypsin solution. It died in 60 hours, and the bacilli were arranged in chains. No. 36 died after 36 hours*. The spleen contained very few bacilli and of these some were arranged in chains.

No. 37 weighed 1865 grammes. It had $2\frac{1}{2}$ cc. of the above solution of trypsin. It was very ill for some days, but at last recovered and is now alive and well nearly three months after the operation. The temperature record of this unfortunately unique case is very interesting. The day after inoculation the temperature was 37°·4 Centigrade, i.e. 2·4 degrees below the normal temperature of a rabbit. It remained at approximately this low figure for some days, shewing a very gradual rise, and only on the sixth day after inoculation had it reached 38°. From this point it rapidly rose till on the 11th day after inoculation it was 40°·1. On the 12th day it stood at 40°·05, when observation of its temperature was discontinued. Another interesting point about the case was the appearance of pus at the seat of inoculation. On the 8th day after the experiment began, a small hard tumour about half-an-inch in diameter was found at the seat of inoculation. On the 13th day a second larger tumour appeared in front of the former. This gradually increased in size and was found to contain caseating pus. About a week later, no further increase in size could be noted. The animal appeared to be ema-

* I noticed while this rabbit was being inoculated that it had a cough. This point is worth noting, as I have usually observed that a lung disorder increases susceptibility to anthrax.

ciated. At the present time it appears to be strong and fat, and only a trace of the swellings can be seen.

Experiment 5. Three rabbits, Nos. 40, 41 and 42, were inoculated with virulent anthrax spores. No. 42 served as control and died within 60 hours. Its spleen contained very few bacilli which were never arranged in chains of more than six joints. No. 40 received five cc. of one per cent. pepsin solution the day after inoculation with anthrax. It died after about 60 hours. The bacilli in the spleen were mostly in chains of 10 to 20 joints. Generally they were arranged in clusters which often surrounded a phagocyte. A few phagocytes containing bacilli were seen.

No. 41 weighed a little over two kilos. It had three cc. of the pepsin solution the day after inoculation, another three cc. were injected after a few hours, and again six cc. on the following day. It died 71 hours after its anthrax inoculation. The spleen was full of bacilli arranged in rows so long as to be difficult to count, reminding one of a gelatine culture. Chains of over 30 segments were noted.

The above experiments appear to be interesting from several points of view, though it must be confessed that they raise more questions than they settle.

In the first place, so far as I know, this is the first time that a substance prepared independently of the pathogenic microbe, and administered after its advent, has been found to exert an influence on the development of the microbe within the body of the animal and so on the course of the disease. Since the ferments that I experimented with have no special known relation to the anthrax bacillus it is to be expected that the same ferments will exert an analogous influence on the course of other diseases, which are similarly produced by pathogenic micro-organisms. Further I have worked with virulent anthrax, a virus which kills 100 per cent. of the rabbits inoculated with it. That is to say, the rabbit shews practically no power of resisting the onset of the malady. May it not be expected that these ferments will shew a still greater power of antagonising the microbe in those diseases and with those animals in which the mortality is only 10 or 20 per cent.?

Secondly, the above results are interesting from the point of view of the phagocyte theory. The upholders of this theory assert that natural or acquired immunity against a disease is due to the greater activity of phagocytes. But by the injection of ferments I have in some cases at any rate endowed the animal with an increased power of resisting anthrax virus. The ferments were injected into the blood plasma, and in this liquid the bacilli lived and were apparently affected independently of any increased phagocyte activity. One of these ferments, the pepsin, is at any

rate a post mortem constituent of most of the animal tissues, and I think these experiments seem to uphold a theory first suggested to me by Dr Lauder Brunton, that the "germicidal power" that the animal body seems to possess is connected with the power it had of producing ferments. This suggestion that I have either increased or closely imitated the natural germicidal power by ferment injection is supported by the symptons exhibited by rabbit No. 37 which finally recovered. As mentioned above, a quantity of pus was found at the seat of inoculation. This is rendered interesting by the fact that a similar effect is produced by inoculating with anthrax an adult rat, an animal which is naturally refractory to this disease. In the case of rabbit No. 37 and of a rat inoculated with anthrax we find a large pus formation at the seat of inoculation; that is to say, not increased activity on the part of the leucocytes but an increased degeneration of these cells. Does it not seem probable that these cells give out certain substances (possibly ferments) which hinder or prevent the growth of the bacilli till at length they can be devoured like any other inert granules by the active phagocytes?

Lastly, these experiments suggest another possibility; namely, that by injecting ferments, other microbes which cannot easily be cultivated outside the body of the animal may be attenuated within it. Possibly in this way attenuated tubercule bacilli could be obtained which might be used as a means of vaccinating against consumption.

NOTE. Since communicating the above results to the Society, I have succeeded in obtaining similar results by injection of Halliburton's cell globulin after inoculation with anthrax. This cell globulin is a proteid, obtained from cells of the lymphatic glands, which is either identical with fibrin-ferment or very closely connected with it. In my experiments with this substance, the elongation of the spleen bacilli was not always so marked as with pepsin and trypsin, but longer chains could always be found in the lymphatic glands near the seat of inoculation. Perhaps more of the individual joints were degenerated than in my former experiments, and in cases where the chains did not consist of many members, the individual joints were often unusually long. I have not yet finished this course of experiments, but the results, as far as they go, support the view I have enunciated in my paper concerning the mechanism of the germicidal power; and the fact that two out of three substances which I have employed are probably constituents of leucocytes appears to me to be particularly suggestive.

December 7, 1889.

November 25, 1889.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

Arthur Berry, M.A., Fellow of King's College.
H. F. Baker, B.A., Fellow of St John's College.
E. W. Brown, B.A., Fellow of Christ's College.
C. Warburton, B.A., Christ's College.

The PRESIDENT announced that the adjudicators of the Hopkins Prize for the period 1880–82 have awarded the Prize to Mr R. T. GLAZEBROOK, F.R.S., for his researches in Physical Optics.

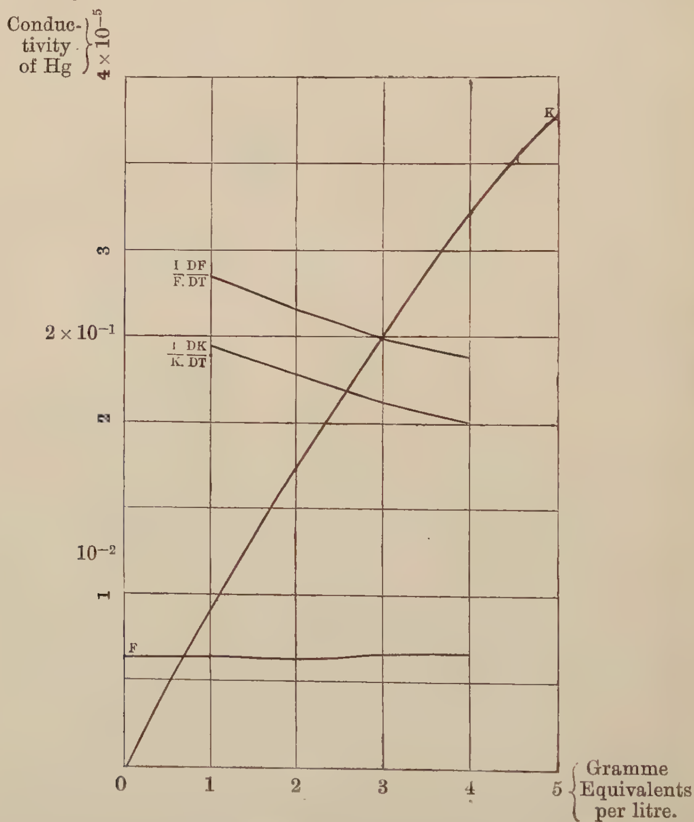
The following Communications were made to the Society:

(1) *On the relation between Viscosity and Conductivity of Electrolytes.* By W. N. SHAW, M.A., Emmanuel College.

IT has long been suspected that the resistance offered by an electrolyte to the passage of electricity through it, depends in some way upon the viscosity of the liquid. The mere fact that the conduction of electricity is in reality convection by moving ions suggests of itself that resistance may be the opposition offered by the fluid to the motion of the ions. It is not however in any way obvious that the resistance which moving ions would meet with would be identical with the ordinary viscosity that has to be overcome when a fluid is driven through a capillary tube; so that when G. Wiedemann, in 1856, found that there was an analogy between the relative magnitudes of the numbers expressing the conductivity of certain solutions and those expressing the fluidity (the reciprocal of the viscosity), the implied relation between electric conductivity and fluidity was not at once accepted as proved. And indeed, in its crudest form, the hypothesis that conductivity is identical with fluidity, as we measure it by Coulomb's method or Poiseuille's method, or that the two are proportional, evidently cannot be maintained. For clearly a salt solution becoming gradually more and more dilute approaches a finite limit of viscosity, namely the viscosity of pure water, whereas the conductivity apparently diminishes without limit. Moreover there are numerous mobile liquids which do not conduct at all. Nor is the mobility which is characteristic of fluids really necessary to electrolytic conduction. Professor W. Kohlrausch has examined the conductivity of fused salts of silver through a wide range of temperature, and he finds that in the case of iodide of silver there is no discontinuous change in the conductivity at the melting-point, on the contrary the resistance only increases very gradually after solidification takes place, remaining less than the minimum

resistance of H_2SO_4, until the mass becomes crystalline and then a sudden increase of resistance occurs. The curve representing the variation of resistance with temperature for the mixed chloride and iodide shews similar properties. Moreover Arrhenius has measured the resistance of electrolytes which contain gelatine, and these solidify without producing any sudden change of resistance. Hence clearly resistance and ordinary viscosity are not the same thing.

Furthermore, in comparing the numerical values of conductivity and fluidity we are met with an obvious contradiction of any such



generalisation; for on the addition of an acid or a salt to water the viscosity may be increased, whereas the conductivity of the solution depends entirely on the presence of the salt or acid. If the curves shewing the variation of conductivity with concentration be compared with those for fluidity and concentration no similarity is conspicuous although there are some striking instances, particularly that of sulphuric acid, of concurrence of pecu-

liarities which arrest attention. As an example of very normal type, I have in fig. 1 reproduced the curve (k) of conductivity and concentration in gramme equivalents per litre of NH_4Cl from Wiedemann's *Electricität*, vol. I. p. 610. And to compare with it I have plotted a curve, from observations (f) of Grotian*, of the fluidity of solutions of NH_4Cl for the same range of concentration. There is no striking resemblance but rather the reverse. In fact, the curves for conductivity and concentration are generally roughly parabolic in shape with a vertex of maximum conductivity, the equations of a series of the curves are given by Wiedemann†, and the curves for fluidity are according to Reyher‡ and Arrhenius§ very exactly represented by the equation $\eta = A^x$, where η is the viscosity relative to water, x the number of gramme equivalents per litre, and A a constant not differing much from unity. Such curves are of course quite different from the conductivity curves, and yet, when tables of conductivities and fluidities of different solutions are compared, the connexion is very striking; and the obvious suggestion is that both fluidity and conductivity are very complex phenomena, but that for a given solution they are both dependent ultimately in some way upon temperature and concentration, and are so related that, other things being unaltered, if the fluidity be increased the conductivity is consequently improved. It therefore seems more hopeful to compare, as Grotian has done, fractional rates of variation of the two quantities rather than the quantities themselves. Grotian has chosen the rate of variation with temperature, and in parallel columns on pp. 949—952 of vol. II. of Wiedemann's *Electricität* are values of $10^4\phi$ and $10^4\kappa$, where $\phi = \left(\frac{df}{dt}\right)_{22} \frac{1}{f_{18}}$ and $\kappa = \left(\frac{dk}{dt}\right)_{22} \frac{1}{k_{18}}$, for different concentrations of solutions of a large number of salts. The suffixes indicate temperatures, f is fluidity relative to pure water at 10° C., k conductivity referred to mercury, and t temperature. A further advantage of this method of comparison is that the units in which the conductivity and fluidity are measured do not affect the result.

In order to get a better general view of the comparison of the two magnitudes ϕ and κ, I have examined the curves plotted from all the results recorded in Wiedemann, and the general parallelism of corresponding curves is very striking. The set of lines representing either one of the properties for the solutions of the different substances traverse the paper at a very great variety of inclination to the axes; some of them are nearly

* *Pogg. Ann.*, vol. CLX., p. 259, 1877.
† *Electricität*, vol. I., p. 600.
‡ *Zeitschrift für Phys. Chem.*, vol. II., p. 744, 1888.
§ *Ib.*, vol. I., p. 285, 1887.

parallel to the axis of abscissae, others are inclined positively to it, others again negatively; but in all cases, with the possible exception of acetic acid, which exhibits irregularity, the corresponding curve for the other property is a curve which though not coincident is very nearly parallel, and generally speaking when there is a change in the inclination of the curve for one property there is a corresponding change of inclination in the curve for the other property. When we add to the general appearance of parallelism for the lines which present no special peculiarity the remarkable parallelism in the exceptional cases of NaHO, the curves for which are both very steeply but equally inclined to the axis, and of H_2SO_4, which furnishes two irregular but still parallel curves each shewing a maximum for the same degree of concentration, the evidence is convincing that there is a real relation not of identity, but of parallelism between these temperature coefficients of the two quantities for different concentrations.

Another means of altering the viscosity without altering the other properties of a solution might be found in adding an inert non-conducting liquid to the solution and thereby altering the solvent. Experiments have been made in this direction by C. Stephan*, who has measured the viscosity and its temperature coefficient for some mixtures of alcohol and water, and the conductivity and its temperature coefficient for dilute solutions of NaCl, KCl, LiCl, NaI, KI in these mixtures. The investigation differs from Grotian's inasmuch as the temperature coefficients of fluidity are not determined for the solutions but only for the solvents, so that while the fluidity temperature coefficient seems, for the alcoholic solvents, again to be of the same order of magnitude and to exceed the conductivity temperature coefficients of the solutions, no precise comparison can be made. Stephan's comparisons of results are mainly concerned with an enquiry as to the constancy of the products of conductivity of the solution and viscosity of the solvent for the different solvents; the constancy is not established though a limiting value is indicated for very dilute solutions.

E. Wiedemann† has moreover compared the conductivities of corresponding solutions of $NaSO_4$ in water and glycerine, but again no numerical relation between conductivity and resistance is exhibited.

The relation between resistance and viscosity seems therefore not to be a simple one, though the relation between the temperature coefficients does seem from Grotian's observations to be comparatively simple. With the view of exhibiting this aspect

* *Wied. Ann.*, vol. XVII., p. 673, 1882.
† *Ib.*, vol. XX., p. 537, 1883.

of the question we may consider some of Grotian's observations a little more closely.

I have reproduced in fig. 1 the curves of temperature coefficients for NH_4Cl plotted from his tables (reducing the abscissae to gramme equivalents per litre) and placed them on the diagram with the curves for fluidity and conductivity already alluded to. The curves for KCl, KBr and KI are strictly analogous, so that the diagram may be taken as exhibiting the comparison for a group of salts which seem to form a special class of solutions with respect to conductivity and fluidity.

Strict parallelism of the plotted curves would correspond of course to a constant difference between the corresponding values of $10^4 \frac{1}{f}\frac{df}{dt}$ and $10^4 \frac{1}{k}\frac{dk}{dt}$. These differences are tabulated below, by interpolation, for the four salts.

Table I.

Concentration, per cent.	NH_4Cl.	KCl.	KBr.	KI.
5	34			
10	29	30·6		35·0
15	29	30·0	28·4	31·0
20	29	30·4	30·0	29·0
25		23·8	28·4	28·4
30			24·0	28·4
35			22·6	28·0
40				27·0
45				23·4
50				20·0

The differences are very nearly the same for all the salts for concentrations between 10 and 20 p.c., but the two values approach one another slightly when the concentration is large. As a simple assumption we may take however that the relation between the temperature coefficients of fluidity and conductivity is one of a constant difference σ, independent of the concentration and of the nature of the salt for this particular group of salts. It may vary with the temperature and with the nature of the solvent, but we will assume for the moment that it is a constant. We then get

$$\frac{1}{f}\frac{df}{dt} = \frac{1}{k}\frac{dk}{dt} + \sigma \text{ ....}(1),$$

where σ is a small quantity (about 30×10^{-4}) independent of

concentration but depending on the solvent and possibly also on temperature.

The physical interpretation of this equation would be that the effect of temperature upon the conductivity of the solution is of a two-fold nature, (1) the conductivity is indirectly *increased* by the increasing fluidity of the liquid and (2) it is *diminished* by some alteration of the properties of the solvent which does not affect the fluidity. In electrolytes at ordinary temperatures the first effect is predominant, but on very great rise of temperature (without secondary alterations of condition) the second effect might become very great compared with the first. Thus if the solution (with the salt) were volatilized, the conductivity of the gaseous mixture might be only a small fraction of the conductivity of the solution though the viscosity might have become much less. Integrating equation (1) with regard to temperature (assuming σ to be constant) we get

$$\frac{f_t}{f_0} = \frac{k_t}{k_0} e^{\sigma(t-t_0)},$$

or

$$k_t = \frac{k_0}{f_0} f_t e^{-\sigma(t-t_0)}. \quad \text{.......................(2).}$$

In equation (2) $\frac{k_0}{f_0}$ expresses the relation between conductivity and fluidity at a standard temperature*. If this relation be a complicated function of the concentration, as it appears to be, there is no reason to infer a general simplicity of relation between k and f from the fact of their having temperature coefficients which are connected by a simple relation. If we were possessed of experimental data that would enable us to refer the properties of one class of electrolytes to concentration and temperature as variables, in a manner somewhat similar to that in which the properties of gases are referred to pressure and volume, further insight into the nature of the relation might be obtained.

The evident relation between viscosity and resistance has not yet been satisfactorily accounted for. The hypothesis that the motion of the ions, if these be atoms, is opposed by frictional resistance which can be measured as viscosity for ordinary motion of the liquid seems to be regarded as dubious, although Kohlrausch† has shewn "that the supposition of mechanical and electro-

* Since this paper was read I have seen a paper of Arrhenius (*Zeitschr. für Phys. Chem.* Band. IV. Heft. 1, July 1889), in which an equation practically identical with (2) is deduced directly from the theory of dissociation. On that hypothesis σ would be the fractional temperature coefficient of dissociation, and being *negative* would imply, as Arrhenius points out, a temperature of maximum conductivity beyond which the temperature coefficient of conductivity would be negative.

† *B. A. Report*, 1886, p. 343; *Wied. Ann.* vol. VI., p. 207, 1879.

lytic frictional resistance of about equal amount allows the finding of an absolute size of molecules which approaches the sizes found by other methods."

But is it necessary to suppose that the individual atoms are moved through the liquid? May the ions not be parts of complex molecules of salt and solvent which have to be dragged through the liquid? Wiedemann* mentions that such a suggestion has already been made but that it is very risky, yet the doctrine of electrolysis of molecular aggregates as opposed to that of dissociated atomic ions has some adherents, and expressions might be quoted from the writings of well-known supporters of the dissociation-theory to shew that they acknowledge the necessity for regarding the electrolytic molecule as complex in some special cases. I think it may be well to recall that certain phenomena may naturally lead to the same view, although I am well aware that these phenomena have been otherwise explained in a manner that is accepted as satisfactory.

The view that electrolysis consists in the convection of electricity by single atoms only, or their chemical representatives, is based upon the resolution of the processes taking place in an electrolytic cell into independent phenomena. Electric endosmose, the unequal dilution of solution at the electrodes and the deposition of ions are all treated separately. Electric endosmose is regarded as the result of the electrification by contact of the boundary layer of the solution in the porous partitions which divide the cell, and the explanation has been regarded as complete since von Helmholtz shewed that the difference of potential of the boundary layer necessary to explain the effect was not more than a few volts. The unequal dilution at the electrodes is explained by the theory of migration of the atomic ions with unequal velocities, and this explanation has received strong confirmation from Kohlrausch's calculation of resistance, based on these atomic ionic velocities and Lodge's experimental verification of the calculation in the case of hydrogen. But, apart from these reasons, we have no direct evidence that the ions are simply atoms, or their chemical representatives. One of the greatest desiderata in electrolysis is the determination of the actual ions in any case of electrolysis, but it does not seem practicable to identify them. Hitherto the tendency has been to assume atomic ions if possible, and yet complex ions cannot be excluded. I have taken the following cases of electrolysis from Wiedemann†, who is himself a strong opponent of the idea of complex molecular decomposition in general.

* See *Electricität*, vol. II., pp. 953, 962.
† *Ib.*, vol. II.

We see therefore that in spite of the present tendency to reduce the electrolytic action to convection by atomic ions if possible, there are many cases in which the ions are aggregations of atoms, if not strictly molecular, and in some cases molecules are associated with a moving atom in electrolysis. It seems therefore not altogether unreasonable to assume that the decomposition of a complex molecule may not be exceptional but the general rule. Let us therefore take the venturesome step of considering that the whole result of electrolysis consists in the separation of complex molecular aggregates of salt and water each into two parts, each part containing one dissociated atom or its

TABLE II.

+ Ion.	Electrolyte.	− Ion.
K	KHO	HO
H, NH_3	NH_4Cl	Cl
K_2	$K_2Cr_2O_7$	$2CrO_3$, O
Na_2	Na_2HPO_4	$\frac{1}{2}(H_2O,\ P_2O_5,\ O_2)$
UO_2	UO_2Cl_2	Cl_2
K_4	$K_4Fe(CN)_6$	$Fe(CN)_2$, $(CN)_4$
Ag	Ag_2CN_2	AgCN, CN
K	$\left.\begin{matrix}Cd\\K\end{matrix}\right\}I_3$	CdI_2, I
Cd	$2(CdI_2)$ (in alcohol)	CdI, I
Cd	$3CdI_2$ (in alcohol)	2CdI, I

chemical representative, and suppose that the complete result of electrolysis, including the transference of liquid known as endosmose, the migrations of ions, and the deposition on the electrodes may be accounted for by this splitting up of the complex molecule into two parts and the transference of the separated parts in opposite directions, so that each separated atom would be loaded with a number of water molecules or salt molecules or both. If this be assumed, the effect of a porous diaphragm in a solution would be not to cause the transference of liquid but to prevent it slipping back again, and the friction against the plug which would prevent the slip back would be very similar in its mathematical expression to the force supposed by von Helmholtz to cause the transference.

The reasoning by which Wiedemann, on p. 592 of vol. II. of his *Electricität*, shews that electric endosmose is independent of migration seems to me to be as follows. The total gain of cation

in the cathode vessels is not the same for different degrees of dilution when there is a porous diaphragm, but if you subtract the amount due to the increase of volume of solution, the amounts are approximately the same; hence if the increase of volume be regarded as an entirely independent phenomenon the cations may be assumed to be the same; viz. atoms of copper, for all the different degrees of dilution. But the same result would be arrived at if we assumed that in the more dilute solutions a greater number of molecules were associated with the atoms. Assuming that of the salt which is decomposed half is taken from the anode vessel and half from the cathode vessel, I have calculated the molecules that must be decomposed to give the required total gain at the cathode which is tabulated for $CuSO_4$ by Wiedemann for solutions of different strengths. They are as follows

TABLE III.

Concentration in grammes of Copper per 100 cc.	Molecule decomposed.
3·3793	$Cu_2(CuSO_4, 5H_2O)/(SO_4)_2$
3·118	$Cu_2(CuSO_4, 6H_2O)/(SO_4)_2$
2·263	$Cu_2(CuSO_4, 8H_2O)/(SO_4)_2$

The molecules which are actually decomposed may be more complex than those given, by combination with molecules of solution, but the association of such molecules with the ions would not affect the ultimate relative distribution of the electrolyte. Moreover all the ions in a specific solution need not be of the same order of complexity.

It would be a natural consequence of this view to suppose that when the solutions became very dilute, the number of molecules of water associated with the moving atoms would be very large, ultimately being proportional to dilution; in that case the electrolysis would depend mainly on the motion of these large molecular aggregates past each other and the resistance would be of the nature of ordinary viscosity. Under these circumstances the number of molecules associated might depend on the number and be independent of the nature of the atoms and the ionic velocities, and the resistance of all electrolytes would tend to the same value as they are known to do in solutions of extreme dilution*.

* F. Kolrausch, *Gegenwärtige Anschauung, &c.*, p. 28.

I have not hitherto dealt with the difficulty raised by the remarkably accurate calculations of resistance of electrolytes from ionic velocities based on the assumption of atomic ions and verified experimentally by Lodge in the case of hydrogen. In the first place however I find it difficult to realise the conception of an extremely large number of small bodies (atoms) moving in opposite directions *through* an aggregation of other small bodies (the solution). There is also an objection to Kohlrausch's theory on the ground that it assumes all the atoms of the dissolved salt to be moving. It seems to me that the conception is easier if we regard the whole of the solution as divided between the dissociated atoms; the effect of the electromotive force would then be to pull the whole of one set of atoms with their associated molecules in one direction and the whole of the other set with their associated molecules in the opposite direction, and we thus get a stress shearing the one set of molecules past the other set; the result will be a relative motion of the atoms carrying their loads and the mean velocity will depend on the electromotive force and the viscosity; but the motion is relative; as to the absolute velocity of each set the velocities in the two directions may be regarded as equal. A particular atom may at one time be a dissociated one moving to meet a partner with this velocity, at another time it may belong to an associated molecule and be travelling with another dissociated atom in the same direction as before or in the opposite; to determine the mean velocity of all the cation atoms for instance, in one cross-section, we must deduct the number of backward steps it takes in the unit of time as part of a molecule associated with an anion atom from the number of forward ones it takes either as a dissociated atom or part of a molecule associated with a cation; this mean velocity will be equivalent to a transference of the atom through the solution. It is this mean rate of transference which must be always the same for the same atom in very dilute solutions, no matter with what other atom it was associated as a salt, and it is this mean velocity which Lodge has measured.

I think, therefore, that it may be possible to frame a general theory of electrolytic action on the basis of a hypothesis of complex molecular aggregates, dissociated in a solution, or separated by the current, into ions consisting of atoms with attached molecules; and such a theory might explain all the various electrolytic phenomena, including migration, endosmose, and the relation between viscosity and resistance. I am well aware that the brief sketch contained in the foregoing paper cannot be regarded in any way as a complete statement of such a theory, and that further application in detail is necessary before the theory can claim to be satisfactory. All that I venture to say is that there

are no crucial experiments, that I am acquainted with, which definitely and finally contradict it, and until such experiments are made it may be well to bear in mind the possibility of such explanations as the theory affords.

(2) *The finite deformation of a thin elastic plate.* By A. E. H. Love, M.A., St John's College.

(3) *A solution of the equations for the equilibrium of elastic solids having an axis of material symmetry, and its application to rotating spheroids.* By C. Chree, M.A., King's College.

[*Abstract.*]

A solution is obtained of that type of the elastic solid equations which contains five elastic constants, answering to those bodies in which the structure is symmetrical round an axis. The solution proceeds in ascending powers of the variables x, y, z. In the expressions for the displacements terms containing powers of the variables below the fourth are retained. Thus the solution, while complete so far as it goes, can solve exactly only certain classes of problems. One of the problems which it can completely solve is that of a spheroid of any eccentricity rotating uniformly about the axis of revolution, this axis being in the direction of the axis of symmetry of the material; and it is to this problem that attention is mainly directed.

The solution obtained for the general case of a rotating spheroid being somewhat complicated, certain special cases are first considered. The first of these cases, that of a very flat oblate spheroid, applies approximately to a thin plate rotating about the normal to its plane through the centre. The second case, that of a very elongated prolate spheroid, applies even more satisfactorily to the non-terminal portions of a rotating cylinder of length great compared to its diameter. In these two cases the material is of the 5-constant type. The third special case is that of uniconstant isotropy in spheroids of every form.

From the light thrown on the question by the results obtained in the third special case it becomes possible to dissolve out of the complicated mathematical expressions for the general case a very considerable amount of information as to the state both of stress and strain throughout the spheroid. The key to this information is supplied by the recognition in every material whether of the 5-constant or of the isotropic type of a "critical" spheroid. The ratio of the "polar" to the "equatorial" diameter of this spheroid depends only on the elastic constants of the material, and is given by a simple expression. The following, which are only a few of

the results obtained, will show the importance of the critical spheroid:—

In any 5-constant or isotropic rotating spheroid one of the principal stresses is everywhere perpendicular to the "meridian" plane, and of the two in the meridian plane the greater makes an obtuse or an acute angle with the perpendicular on the "polar" axis produced outwards according as the spheroid is more or less oblate than the critical spheroid. In particular the surface is under a tangential tension in the meridian plane in the former case, but under a compression in the latter. In the critical spheroid one of the principal stresses is everywhere zero, and on the surface there is no stress at all in the meridian plane. In any species of bi-constant isotropic material, for a given value of the equatorial diameter, the critical spheroid is the form in which the "tendency to rupture" on Saint-Venant's theory is the greatest.

In the case of uniconstant isotropy the character of the strain throughout rotating spheroids of all shapes is completely investigated, and is shewn in a table. In the general case of 5-constant material a similar, though not so exhaustive, analysis is given. Tables supply the values of the changes in the lengths of the equatorial and polar diameters, and the strains at the centres for various kinds of biconstant isotropic materials in spheroids of various forms.

The variations of some of the more important quantities are also shewn graphically.

(4) *On the concomitants of three ternary quadrics.* By H. F. Baker, B.A., St John's College.

[*Abstract.*]

The author applies a modification of the symbolical method suggested by Clebsch and Gordan (*Math. Annal.* I. 90 and I. 359) to obtain the set of concomitants in terms of which all the system are expressible as rational integral algebraic functions. The result is given by the following table. The forms are taken respectively to be

$$a_x^2 = a_x'^2 = a_x''^2 = \ldots;$$
$$b_x^2 = b_x'^2 = b_x''^2 = \ldots;$$
$$c_x^2 = c_x'^2 = c_x''^2 = \ldots.$$

Also $(aa'u)^2$ is abbreviated into $u_\alpha^2 = u_{\alpha'}^2 = u_{\alpha''}^2 = \ldots;$

and $(aa'a'')^2$ into a_α^2 etc.;

and so for the other two forms.

Such a symbol as (523) preceding a form indicates that the form is of the fifth degree, second class, and third order.

Only one form of a given type is written down; the others may be obtained by interchanging the letters—the number of forms so obtainable is given by the number in brackets which follows. The forms are arranged in sets, as obtained, according to their degrees.

0. $(011) = u_x$ (1)

1. $(102) = a_x^2$ (3)

2. $(212) = (bcu)\, b_x c_x$ (3)
$(220)_1 = u_\alpha^2$ (3)
$(220)_2 = (bcu)^2$ (3)

4. $(410) = (bcu)\, b_\alpha c_\alpha$ (3)
$(402)_1 = b_\alpha c_\alpha b_x c_x$ (3)
$(402)_2 = (\beta\gamma x)^2$ (3)
$(421)_1 = (bcu)\, b_\alpha c_x u_\alpha$ (6)
$(421)_2 = (bcu)\, b_\gamma c_x u_\gamma$ (6)
$[421]_3 = (a'bc)(uca)(uab)\, a_x'$ (3)
$(421)_4 = (\beta\gamma x)\, u_\beta u_\gamma$ (3)

6. $(600) = (\alpha\beta\gamma)^2$ (1)
$(611)_1 = (\alpha\beta\gamma)(\beta\gamma x)\, u_\alpha$ (3)
$[611]_2 = a_\beta a_\gamma b_\gamma b_x u_\beta$ (6)
$[630]_1 = (\alpha\beta\gamma)\, u_\alpha u_\beta u_\gamma$ (1)
$[630]_2 = (abu) a_\beta b_\gamma u_\beta u_\gamma$ (6)
$[630]_3 = (bcu)\, u_\beta u_\gamma b_\gamma c_\beta$ (3)
$(603)_1 = (\beta\gamma x)(\gamma\alpha x)(\alpha\beta x)$ (1)
$[603]_2 = (\beta\gamma x)\, a_x b_x a_\beta b_\gamma$ (6)
$[603]_3 = (\beta\gamma x)\, b_x c_x b_\gamma c_\beta$ (3)

3. $(300)_1 = a_\alpha^2$ (3)
$(300)_2 = b_\alpha^2$ (6)
$(300)_3 = (abc)^2$ (1)
$(311)_1 = u_\alpha b_\alpha b_x$ (6)
$(311)_2 = (abc)(bcu)\, a_x$ (3)
$[303] = (abc)\, a_x b_x c_x$ (1)
$(330) = (bcu)(cau)(abu)$ (1)

5. $[501]_1 = (abc)\, a_x b_\alpha c_\alpha$ (3)
$(501)_2 = (\beta\gamma x)\, a_\beta a_\gamma$ (3)
$(520) = u_\beta u_\gamma a_\beta a_\gamma$ (3)
$(512)_1 = (\beta\gamma x)\, a_\beta a_x u_\gamma$ (6)
$[512]_2 = (abc) a_\beta u_\beta b_x c_x$ (6)
$(512)_3 = (\beta\gamma x)\, c_x c_\beta u_\gamma$ (6)

7. $[710]_1 = (\alpha\beta\gamma) a_\beta a_\gamma u_\alpha$ (3)
$[710]_2 = (bcu)\, a_\beta a_\gamma b_\gamma c_\beta$ (3)
$[721] = (\alpha\beta\gamma)\, b_\alpha b_x u_\beta u_\gamma$ (6)

8. $[801]_1 = (\beta\gamma x)\, b_\gamma c_\beta b_\alpha c_\alpha$ (3)
$[801]_2 = (a'bc)\, a_\beta a_\gamma b_\gamma c_\beta a_x'$ (3)
$[812] = (\alpha'\beta\gamma)(\gamma\alpha x)(\alpha\beta x) u_{\alpha'}$ (3)

9. $[911] = a_\beta a_\gamma b_\gamma b_\alpha c_\alpha c_x u_\beta$ (6)

$[1010] = (\alpha'\beta\gamma)\, b_\gamma c_\beta b_\alpha c_\alpha u_{\alpha'}$ (3)

The degree—class—order symbols of eighteen of the types are placed in square brackets. This indicates that they are reducible after multiplication by u_x. Some of them are further reducible on multiplication by u_x^2: namely these are $(501)_1$; $(710)_1$; $(801)_2$; (911); (1010). Allowing these reductions the system is expressible by 13 kinds of forms, *viz.* by

$$\left\{\begin{array}{llllllll} (abc)^2 & b_\alpha^2 & a_\alpha^2 & a_x^2 & (bcu)\, b_x c_x & (bcu)^2 & u_\alpha b_\alpha b_x & (abc)(bcu)\, a_x \\ (\alpha\beta\gamma)^2 & & & u_\alpha^2 & (\beta\gamma x)\, u_\beta u_\gamma & (\beta\gamma x)^2 & & (\alpha\beta\gamma)(\beta\gamma x)\, u_\alpha \end{array}\right.$$

$$\left\{\begin{array}{lllll} (bcu)(cau)(abu) & (bcu)\, b_\alpha c_\alpha & b_\alpha c_\alpha b_x c_x & (bcu)\, b_\alpha c_x u_\alpha & (bcu) b_\gamma c_x u_\gamma \\ (\beta\gamma x)(\alpha\beta x)(\gamma\alpha x) & (\beta\gamma x) a_\beta a_\gamma & u_\beta u_\gamma a_\beta a_\gamma & (\beta\gamma x)\, a_\beta a_x u_\gamma & (\beta\gamma x)\, c_x c_\beta u_\beta \end{array}\right.$$

where a form and its reciprocal, as well as forms of the same type, are counted as being of the same *kind*.

Note. There are often identical integral relations among forms of the same type. These are not here set down. Further, if we allow algebraic functions of any rational kind, all the simultaneous concomitants can be expressed in terms of fifteen concomitants (Forsyth, *American Journal of Mathematics*, XII. p. 54).

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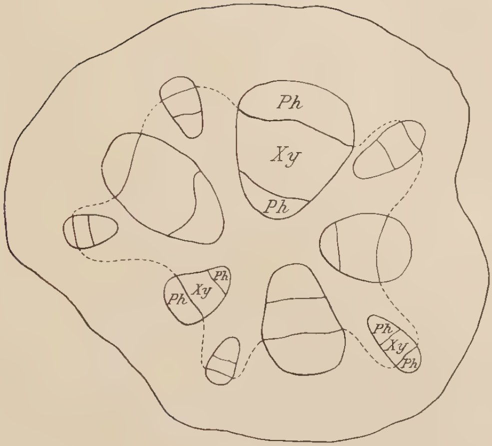

Fig. 1

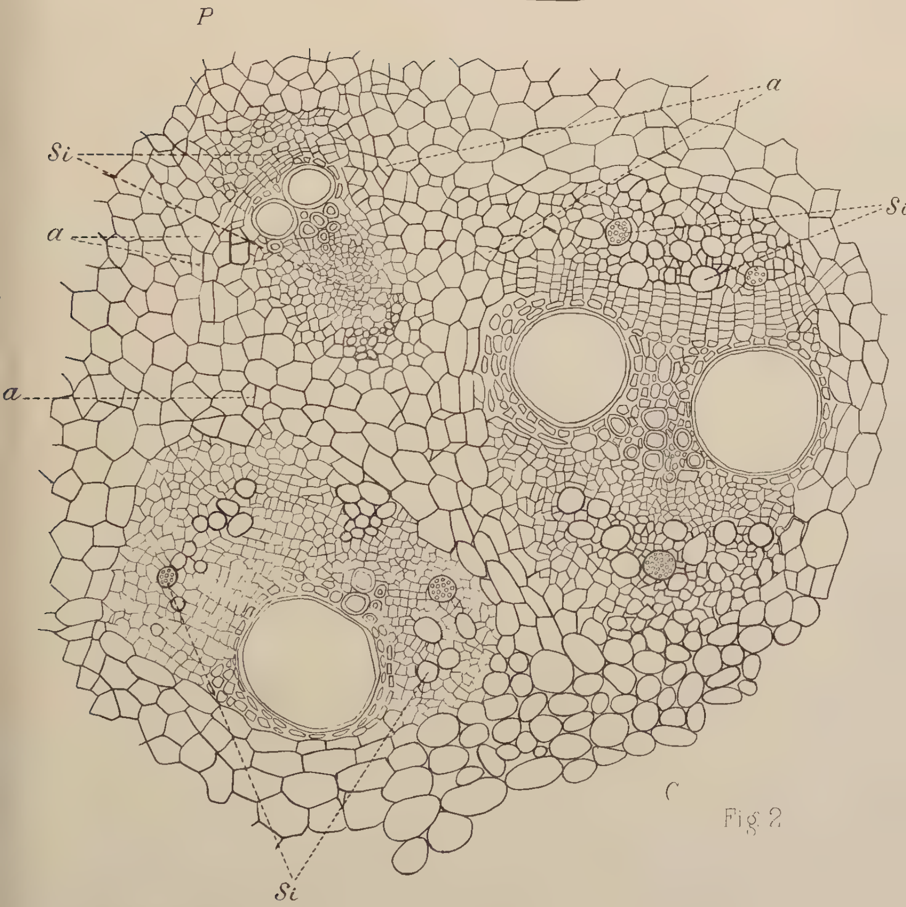

Fig. 2

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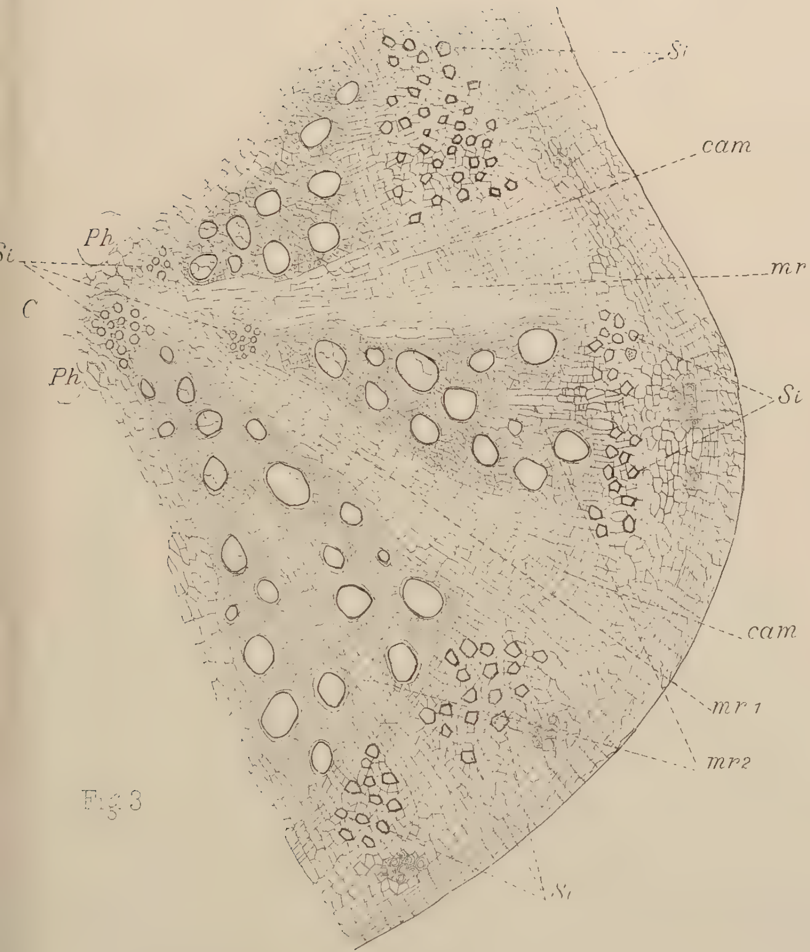

Fig. 3

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PROCEEDINGS

OF THE

Cambridge Philosophical Society.

January 27, 1890.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

J. G. Adami, M.A., M.B., Christ's College.
T. Roberts, M.A., St John's College.
R. H. Solly, M.A., Downing College.
E. H. Hankin, B.A., St John's College.

The PRESIDENT called attention to the proposed International Memorial to Dr Joule, late Honorary Fellow of the Society, and announced that subscriptions might be sent to the Honorary Secretary, Joule Memorial, Royal Society of London, or that they would be received by Mr R. T. Glazebrook, Treasurer of the Cambridge Philosophical Society.

The following Communications were made to the Society:

(1) *Non-Euclidian Geometry.* By Prof. CAYLEY.

(*Abstract.*)

The chief object of the Memoir is the development of the analytical theory: and as the form assumed for the equation of the Absolute is $x^2 + y^2 + z^2 + w^2 = 0$, the formulæ obtained may be regarded as belonging to Elliptic Space. But this is not the point of view of the Memoir; the space considered is ordinary space, it is only the notion of distance (linear, angular, and dihedral) which is modified. Thus for instance, lines perpendicular to each other,

in the ordinary sense, exist, but there is no occasion to consider them: in place thereof we consider lines which are in the new sense perpendicular to each other, and the theory is an entirely distinct one; given any two lines, we have perpendicular to each of them (not a single line, but) two lines, or say there are two perpendicular distances: the theory of these distances is considered in some detail.

(2) *A Scheme of the Simultaneous Motions of a system of Rigidly connected Points, and the Curvatures of their Trajectories.* By J. LARMOR, M.A., St John's College.

The following analysis is suggested by the theorems of De la Hire and Savary, whereby the determination of the curvatures of the trajectories of the different points of a solid moving in one plane is reduced to geometrical construction. In this theory the construction is based on the circle which at the instant in question is the locus of points for which the curvature is zero, the well-known circle of inflexions. See Williamson's *Differential Calculus*, Chapter XIX.*

In the generalized theory, when the motion of the solid is not confined to be uniplanar, the first problem is to determine the nature of the locus of inflexions. This is easily effected by kinematical considerations; for the criterion of a point x, y, z being on the locus is that its acceleration is in the same direction as its velocity, viz. that

$$\frac{\ddot{x}}{\dot{x}} = \frac{\ddot{y}}{\dot{y}} = \frac{\ddot{z}}{\dot{z}} \qquad\qquad (1).$$

Now we may specify the motion of the solid by u, v, w the components of the velocity of the origin, and ω_x, ω_y, ω_z the component angular velocities of the body round the axes of coordinates. Then, as usual,

$$\dot{x} = u - y\omega_z + z\omega_y \qquad\qquad (2),$$

$$\ddot{x} = \dot{u} - y\dot{\omega}_z + z\dot{\omega}_y - \omega_z(v - z\omega_x + x\omega_z) + \omega_y(w - x\omega_y + y\omega_x) \qquad\qquad (3),$$

with two pairs of other similar formulae.

The equations of the curve of inflexions are now obtained by substitution in (1).

* I find that questions similar to the ones here discussed are analyzed by the method of vectors from a fixed origin in the *Comptes Rendus*, 1888, pp. 162—5, by Gilbert, who also gives references to other writers on this subject. His investigations relate chiefly to the case when a point of the system is fixed.

The principal results obtained in this note have been stated in the *Cambridge Mathematical Tripos, Part II.*, June 1, 1889. (*Camb. Exam. Papers*, 1888–9, p. 569.)

The result will be simplified if we take the central axis of the motion for the axis of x, so that $u = V$, $\omega_x = \Omega$, while the other components vanish, though their fluxions remain finite. We thus obtain the equations

$$\frac{L}{\varpi} = \frac{M - \Omega^2 y}{z} = \frac{N - \Omega^2 z}{-y} \quad \text{.....................(4)},$$

wherein

$$\left.\begin{aligned} L &= \dot{u} - y\dot{\omega}_z + z\dot{\omega}_y \\ M &= \dot{v} - z\dot{\omega}_x + x\dot{\omega}_z \\ N &= \dot{w} - x\dot{\omega}_y + y\dot{\omega}_x \end{aligned}\right\} \quad \text{.......................(5)},$$

and ϖ is written for V/Ω, the pitch of the given screw-motion.

The equations (4) thus obtained may readily be verified by intuition; for L, M, N represent the component accelerations due to the motion of the origin and the change of values of the angular velocities, while 0, $-\Omega^2 y$, $-\Omega^2 z$ are the components of the centrifugal force round the central axis, and it is clear that these together make up the total acceleration.

These equations (4) represent the curve of intersection of two paraboloids. To reduce them to the simplest possible form, first turn the axes of y, z round that of x, so as to make $\dot{\omega}_z$ zero. Then move the origin along the central axis a distance h, so that the equation referred to this new origin is obtained by writing $x + h$ in place of x, and take $h = \dot{w}/\dot{\omega}_y$. We thus have finally

$$\frac{\dot{u} + z\dot{\omega}_y}{\varpi} = \frac{\dot{v} - z\dot{\omega}_x - \Omega^2 y}{z} = \frac{-x\dot{\omega}_y + y\dot{\omega}_x - \Omega^2 x}{-y} \quad \text{.......(6)},$$

where we notice by the way that the numerators give the simplest form to which the rectangular component accelerations for a moving solid can be reduced.

The equations (6) represent the curve of intersection of a parabolic cylinder having its generators parallel to the axis of x with a rectangular-hyperbolic paraboloid having its axis in the same direction. These surfaces intersect on the plane infinity along the line where any plane z = constant meets it. The finite part of their curve of intersection, which is the proper inflexional curve, is therefore a *twisted cubic.* Its equations (6) may be put in the form

$$\left.\begin{aligned} y &= A + Bz(\alpha + \beta z) \\ x &= y(\alpha + \beta z) \end{aligned}\right\} \quad \text{.......................(7)},$$

which are unicursal in the parameter z.

We remark that a wire of this form is the most general solid that can be moved with rotation so that all its points are instantaneously describing straight paths; also that any wire whose

form is given by (7) possesses this property, the movement being a screw of pitch $-\beta^{-1}(1+B^{-1})$ round the axis of x, eased off in a way that retains one degree of indeterminateness.

We proceed to investigate the trajectory of any point of the solid by the aid of this cubic.

Through any point, as is well known, one and only one chord of the cubic can be drawn. We may regard this chord as a line of constant length moving with its extremities on two fixed lines, which may be considered *straight* so far as the determination of accelerations and curvatures is concerned.

Consider two consecutive positions of it, BC and $B'C'$; let $BP = B'P' = \rho$, and $CP = C'P' = \rho'$, and let $\rho + \rho' = a$. Complete

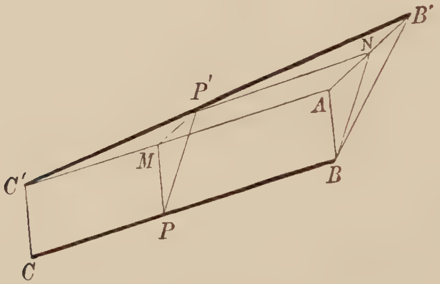


Fig. 1.

the parallelogram $C'CBA$, and draw $P'M$, $P'N$ parallel to AB', AC', as in Fig. 1. The circumstances of the motion are given by the velocities of the extremities of BC; let then

$$\left.\begin{aligned} BB' &= bt + \tfrac{1}{2}\dot{b}t^2 \\ CC' &= ct + \tfrac{1}{2}\dot{c}t^2 \end{aligned}\right\} \quad \text{.........................(8)},$$

so that b, c are the velocities, and $\dot{b}$, $\dot{c}$ the accelerations of B and C along their straight trajectories.

The point Q moves in a fixed plane which is parallel to both BB' and CC', being parallel to the plane ABB'. The coordinates of Q referred to axes of x and y parallel to BB' and CC' are the same as the coordinates of N referred to axes BB' and BA. They are therefore given by

$$\left.\begin{aligned} BB' &= \frac{a}{\rho}x = bt + \tfrac{1}{2}\dot{b}t^2 \\ CC' &= \frac{\rho}{a}y = ct + \tfrac{1}{2}\dot{c}t^2 \end{aligned}\right\} \quad \text{....................(9)}.$$

These are the equations of the path of P, correct as far as the second order, and referred to the parameter t.

The determination of the radius of curvature R at the origin may now be made by the usual methods, bearing in mind that the angle between the axes is ω, the angle between BB' and CC'. It is sufficient to give the result

$$R = \frac{(b^2\rho'^2 + c^2\rho^2 + 2bc\rho\rho' \cos\omega)^{\frac{3}{2}}}{a \sin\omega\,(b\dot{c} - \dot{b}c)\,\rho\rho'} \quad \ldots\ldots\ldots\ldots(10),$$

for its interpretation suggests a purely geometrical method of arriving at it, as follows.

The velocity V of P' is the same as that of N, and is therefore the resultant of velocities $\frac{\rho'}{a}b$ and $\frac{\rho}{a}c$ parallel to BB' and BA; thus

$$V^2 = \frac{1}{a^2}(b^2\rho'^2 + c^2\rho^2 + 2bc\rho\rho' \cos\omega) \quad \ldots\ldots\ldots(11).$$

In the same way, the acceleration f of P' is the resultant of accelerations $\frac{\rho'}{a}\dot{b}$ and $\frac{\rho}{a}\dot{c}$ parallel to BB' and BA; its value may therefore be written down.

Also, if θ denote the angle between V and f, we have $Vf\sin\theta$ equal to the area of the parallelogram contained by the vectors representing V and f; therefore

$$Vf\sin\theta = \left(\frac{\rho'}{a}\,b\,\frac{\rho}{a}\,\dot{c} - \frac{\rho}{a}\,c\,\frac{\rho'}{a}\,\dot{b}\right)\sin\omega$$

$$= \frac{\rho\rho'}{a^2}(b\dot{c} - \dot{b}c)\sin\omega \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

Now by Huygens' fundamental formula of centripetal acceleration in a curve,

$$f\sin\theta = \frac{V^2}{R} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13),$$

therefore we arrive at the formula (10), which may also be written

$$R = \frac{a^2}{(b\dot{c} - \dot{b}c)\sin\omega}\cdot\frac{V^3}{\rho\rho'} \quad \ldots\ldots\ldots\ldots\ldots(14).$$

We may exhibit the result in a geometrical form.

Draw $O\beta$, $O\gamma$ to represent the velocities of B and C in magnitude and direction; divide the fixed line $\beta\gamma$ in ν so that $\beta\nu$ is to $\nu\gamma$ as BP to CP. Then $O\nu$ represents the velocity of P in magnitude and direction, and

$$R = \kappa \frac{O\nu^3}{\beta\nu \cdot \gamma\nu} \qquad \text{.................(15)},$$

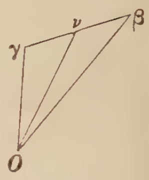


Fig. 2.

where κ represents the constant factor

$$\frac{\beta\gamma^2}{(b\dot{c} - c\dot{b}) \sin \omega}.$$

It is worthy of notice that altering the accelerations of B or C only alters the value of the factor κ; so that the curvatures of the trajectories of all points on BC are altered in the same ratio.

We have thus from Fig. 2 a complete specification of the velocity and curvature for any point P on BC; for the plane of the trajectory is that parallel to the directions of motion of B and C. The acceleration of P may be constructed from those of B and C in the same way as the velocity. The value of the curvature involves the accelerations of B and C only as entering into κ. A known value of the curvature at any one point determines κ once for all, and the values of these accelerations are no longer necessary. A geometrical form for κ is given by (21).

The construction of Fig. 2 applies to any line in the moving solids in so far as the determination of velocities only is concerned; for the determination of accelerations and curvatures it applies only to a chord of the curve of inflexions, as above.

If however the planes of the curvatures at any two points on the line are parallel, it must be such a chord. This may be established by an easy extension of the method of Fig. 1. For taking B, C to represent these two points, we have now BB' and also CC' and BA curved instead of straight lines, and we obtain a quadratic equation giving two positions of N on AB' for which the curvature of the path of P is zero; this gives two points, real or imaginary, on BC, which are also on the curve of inflexions.

The formula (10) for the curvature should be reducible to a form depending only on the geometry of the diagram. In fact, if
$$BB' = h,\ CC' = k,\ B'BC = \beta,\ C'CB = \gamma,$$
we have

$$B'C'^2 = a^2 + h^2 + k^2 - 2ah \cos \beta - 2ak \cos \gamma - 2hk \cos \omega,$$

so that, as $B'C' = a$, h is determined in terms of k by the equation

$$h^2 - 2h(a \cos \beta + k \cos \omega) - 2ak \cos \gamma + k^2 = 0 \quad \text{......(16)},$$

which gives, to the first order,

$$h = -k\frac{\cos\gamma}{\cos\beta'} \quad\text{.........................(17)},$$

to the second order,

$$h = \frac{-1}{2a\cos\beta}\left(1-\frac{k\cos\omega}{a\cos\beta}\right)\left(2ak\cos\gamma - k^2 - k^2\frac{\cos^2\gamma}{\cos^2\beta}\right)$$

$$= -k\frac{\cos\gamma}{\cos\beta} + \frac{k^2}{2a\cos^3\beta}(\cos^2\beta + \cos^2\gamma + 2\cos\beta\cos\gamma\cos\omega)\text{...(18)}.$$

Now by (8) we have to the same order

$$h = \frac{b}{c}k + \frac{1}{2}\left(\dot{b} - \frac{b\dot{c}}{c}\right)t^2$$

$$= \frac{b}{c}k + \frac{\dot{b}c - b\dot{c}}{2c^3}k^2 \quad\text{.......................(19)}.$$

Comparing these, we have

$$b\cos\beta = -c\cos\gamma\text{.......................(20)},$$

$$b\dot{c} - \dot{b}c = \frac{c^3}{a\cos^3\beta}(\cos^2\beta + \cos^2\gamma + 2\cos\beta\cos\gamma\cos\omega)\text{...(21)}.$$

Substituting in (10),

$$R = \frac{(\rho^2\cos^2\beta + \rho'^2\cos^2\gamma - 2\rho\rho'\cos\beta\cos\gamma\cos\omega)^{\frac{3}{2}}}{\rho\rho'\sin\omega\,(\cos^2\beta + \cos^2\gamma + 2\cos\beta\cos\gamma\cos\omega)}\text{...(22)},$$

where β and γ are the angles which BC makes with the curve of inflexions at B and C, and ω is the angle between the tangents to the curve at those points; so that the curvature is expressed in terms of purely geometrical quantities.

It is easy to verify that this theory leads to the correct results for uniplanar motion. For in (4) we have now $\varpi = 0$, $\dot{\omega}_z = 0$, $\dot{\omega}_y = 0$; the curve of inflexions is therefore a *circle* passing through the central axis I. If now the special chord PI meet this circle again in Q, we may apply (14) if we write in it $b = 0$, $\dot{b} = \omega^2 IC$; thus we obtain (and still more directly from (22)) the correct result

$$R\,.\,PQ = IP^2\text{.......................(23)}.$$

The theory for uniplanar motion may also be reduced to simple kinematic considerations as follows. There is one point connected with the solid which has no acceleration; by taking this point for origin and reducing it to rest in the usual manner, we see that the acceleration of any other point with respect to it,

i.e. in this case the total acceleration, is made up of a component $r\dot{\omega}$ transverse to the radius vector and a component $r\omega^2$ towards the origin. Therefore the resultant acceleration is

$$(\omega^4 + \dot{\omega}^2)^{\frac{1}{2}} r,$$

and it acts at a constant inclination α to the radius vector, which is given by $\tan\alpha = \dot{\omega}/\omega^2$; a known theorem. Thus if I denote the instantaneous centre of the motion, I' this centre of accelerations, and P any point, so that

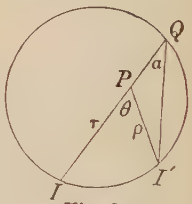


Fig. 3.

$$IP = r, \quad I'P = \rho, \quad I\rho I' = \theta,$$

we have, by the theorem of centripetal acceleration,

$$(\omega^4 + \dot{\omega}^2)^{\frac{1}{2}} \rho \sin(\theta - \alpha) = \frac{\omega^2 r^2}{R},$$

therefore
$$R = \frac{r^2 \sin\alpha}{\rho \sin(\theta - \alpha)}.$$

The points whose paths have zero curvature are given by $\theta = \alpha$, and therefore lie on a circle through I and I', the circle of inflexions.

Let this circle cut PI in Q; then

$$\rho \sin(\theta - \alpha) = PQ \sin\alpha,$$

therefore
$$R = IP^2/PQ.$$

February 10, 1890.

PROFESSOR BABINGTON, VICE-PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:

(1) *On the perceptions and modes of feeding of fishes.* By W. BATESON, M.A., St John's College.

In the course of observations made at Plymouth and elsewhere it appeared that the majority of Fishes are diurnal in their habits and seek their food by sight, but that a minority are almost entirely nocturnal and hunt by scent. To the latter class belong *Protopterus*, Skates and Rays, the Rough Dogfish, Sterlet, Eel, Conger, Rocklings, Loaches, Soles, &c. These creatures remain buried or hidden by day but career about at night in search of food, returning to their own places at dawn. If while they are

thus lying hid, food or even the juice of food-substances is put into the water, they come out after an interval and search vaguely, without regard to the direction whence the scent proceeds. Some of the animals (Rocklings, Sterlet) have special tactile organs in the shape of barbels or filamentous fins with which they investigate their neighbourhood, while others (Conger and Eels) feel about with their noses. None of the fishes which hunt by scent seem able to recognise food by the sense of sight, even though it be hanging freely before their eyes.

The mode of feeding of the Sole is peculiar. When searching for food its skin is more or less covered with sand, which renders it inconspicuous when moving on the bottom. This sand adheres to mucus which is probably exuded when the smell of food is perceived. The Sole seeks its food exclusively on the bottom, creeping about and feeling for it with the lower side of its face. If a worm is lowered by a thread until it actually touches the upper side of the head of a Sole, the animal is still unable to find it but continues to feel for it on the sand. There is however no reason to suppose that the sight of these fishes is deficient. A Rockling at Plymouth had already learnt to come out to be fed if any one came near the tank, though it still did not recognise a worm swimming in the water. Particulars were given of the various irideal mechanisms which occur among fishes.

This investigation was undertaken at the instance of the Marine Biological Association as a preliminary step towards improving the supply of bait. The experience gained suggests that a bait for the south coast, where Conger and Skate are chiefly caught, could be made by extracting the flavour of Squid or Pilchard and compounding it with a suitable ground-substance. Though few practical experiments were made, it was found that an ethereal extract of *Nereis* or Herring, for example, greatly attracted some of these fishes.

(2) *Notes on* Lomatophloios macrolepidotus (Goldg.). By A. C. Seward, M.A., St John's College.

[*Received January* 25, 1890.]

In the *Zeitschrift der deutschen geologischen Gesellschaft* (Band xxxiii. p. 354) Prof. Weiss of Berlin gives a short account of an interesting fossil plant from Langendreer in the Westphalian coalfield: it is preserved in Siderite ("Spatheisenstein") and, according to Weiss, is a cone-like specimen having the characteristic external features of *Lomatophloios macrolepidotus* (Goldg.); size of specimen 18 cm. long and 13 cm. broad, with a fairly uniform thickness of about 3·5 cm.[1]

[1] The form of the specimen and the character of the leaf-bases are shewn in figs. 3 and 4 (Plate III.).

From an examination of microscopical sections Weiss came to the conclusion that the structure was that of a fruit-cone of peculiar organisation. The internal structure is thus described[1]: "From an axis of considerable breadth (about 12 cm.) proceed the lower parts of the leaf organs which have the well-known rhombic leaf-bases with transverse leaf-scars at their upper end. The leaf-bases are sack-like and arched towards the lower end, from which they curve upwards and outwards: in their lower part they are much inflated, and almost reach to the leaf-base next above; they then gradually become narrower until they are of the same breadth as the leaf-scars; this inflated lower portion encloses a sack- or flask-shaped space. The organisation can be easily made out as the tissue is in part well preserved. The flask-shaped spaces contain large (sometimes 2·5 mm. in diameter) round and elliptical bodies, sections of which shew them to be bounded on the outside by a wall formed of polygonal cells and enclosing in their interior numerous grains." These round and elliptical bodies are considered by Weiss to be sporangia full of spores. After this description Weiss goes on to say that the specimen must be regarded as a fruit-cone of *Lomatophloios* with an internal structure comparable to that of *Isoëtes.* The same specimen is also briefly described in the *Botan. Centralblatt*[2]. In his small book on the coal-measure flora[3] Weiss gives a figure of *Lomatophloios macrolepidotus,* which is described in the text as a large cone under the name of *Lepidostrobus macrolepidotus.* Solms-Laubach[4] refers to the description of the same fossil, as given by Weiss, and, judging from the extraordinary size of the axis, throws out the suggestion that in this case we probably have a fructification which was borne on the leaves of the main stem, and not on a fruit-bearing branch. Schenk[5] mentions *Lepidophloios macrolepidotus* (Weiss) as an example of a cone with a very stout axis, and probably belonging to *Lepidophloios.* Prof. Weiss has unfortunately never published any figures to illustrate his description of the internal structure of this so-called *Lomatophloios* or *Lepidodendron* fructification. When recently going through the collection of fossil plants in the Berlin Bergakademie I saw the specimen and also had an opportunity of examining microscopically some of the prepared sections[6].

The microscopic characters are briefly noticed by Weiss: one or two additional points may however be mentioned. On each of the leaf-bases there is a small indentation (Plate III. fig. 4, *i*)

[1] *Zeitschrift der deutschen geol. Gesell.*, Bd. XXXIII. p. 355.
[2] *Botan. Centralblatt.*, Vol. VIII. p. 157.
[3] *Flora der Steinkohlenformation*, Fig. 33, and p. 7.
[4] *Einleitung in die Paläophytologie*, p. 241.
[5] *Die fossilen Pflanzenreste*, p. 70.
[6] I had access to the specimens through the kindness of Prof. Weiss.

immediately below the leaf-scars (fig. 4, *s*); on the leaf-scars I did not notice any definite traces of vascular bundle-scars such as are represented in Weiss' figure[1] (this figure is not taken from the specimen in the Bergakademie, but seems to be copied from the one given by Goldenberg)[2]; the absence of traces of these bundle-scars is probably accidental, and due to imperfect preservation. The lower parts of the leaf-cushions have a more or less wrinkled appearance, in fact they remind one strongly of the persistent petiole bases on the stem of *Cycas revoluta.*

Fig. 1 (Plate III.) represents a section taken from Weiss' specimen. Here we see the form of the leaf-bases as described by Weiss; in the interior of the spaces which occur in the lower part of the leaf-bases are several Stigmarian rootlets with the central vascular bundles. These appear to be what Weiss took for sporangia; that they are in reality Stigmarian rootlets I have not the least doubt; they correspond exactly to those figured by Williamson[3].

As is usual with these rootlets, the peripheral cortical layer of parenchymatous tissue is preserved; also the vascular bundle, the latter being sometimes surrounded by a delicate ring of parenchymatous tissue. Anyone who has examined a number of sections of coal measure plants cannot fail to be familiar with these ubiquitous rootlets. Various observers have fallen into the error of mistaking these intruded rootlets of *Stigmaria* for tissues of the plants into which they happened to have bored their way and with which they are in no way organically connected. Prof. Williamson[4] refers to such mistakes made by Prof. Goeppert, who described in his *Genres des Plantes fossiles* a *Stigmaria* with bundles in the pith: the same mistake was afterwards made by Sir Joseph Hooker and Mr Binney[5]. Solms-Laubach[6] speaks of the ubiquitous nature of Stigmarian "appendices" and reproduces one of Renault's figures as an illustration.

In fig. 1 (Plate III.) eight Stigmarian rootlets are seen.

In fig. 2, the cortical tissues are shewn considerably magnified. At *A*, the section shews prosenchymatous cells more or less compressed and not very clearly defined; as we pass along the section we come to other prosenchymatous cells in a better state of preservation, farther on these become shorter and less distinct; at *B*, the cortical tissues end: after a short break we come to the leaf tissue at *C*, where the cells appear to be parenchymatous; as

[1] *Flora der Steinkohlenformation*, Fig. 33.

[2] *Flora Saraeponta*, Pl. XIV.

[3] *A Monograph on the Morphology and Histology of Stigmaria ficoides* (Palæontographical Society, London, 1887), Pl. x. Fig. 42.

[4] *Ibid.* p. 13.

[5] *Q. J. G. S.*, Vol. xv. p. 76.

[6] Solms-Laubach, *loc. cit.* p. 294.

they approach the periphery *D* they become somewhat oblique and gradually get more and more compressed. From *E* to *F* we have similar tissues belonging to another leaf-base.

Williamson[1] gives a figure of *Favularia* shewing the "tubular part of the bark[2]" passing through prosenchymatous cells into the outer parenchymatous tissue of the epidermal layer. In my fig. 2, in passing from *A* to *B*, almost identically the same tissues are seen. From *C* to *D* and *E* to *F* we have parts of the parenchymatous epidermal layer which has passed off to form the leaf-bases. The cells in this tissue become smaller and denser towards the periphery, as noticed in similar sections by Williamson[3].

At *a* in fig. 1 are some opaque bodies, round and elliptical in shape (too small to be shewn in the figure). These I took to be coprolites of some wood-boring Annelid; such bodies are by no means uncommon in sections of fossil wood; in one of Williamson's figures of a Calamite[4] some of these coprolites are seen in its medullary cavity, in another place[5] a piece of wood is shewn perforated by some xylophagous animal whose coprolites occur scattered about the tissues. These coprolites and the vascular bundles of the Stigmarian rootlets seen in transverse section seemed to me the only things that Weiss could have taken for spores.

The conclusion to which an examination of the above sections led me was that this so-called cone of fructification is simply a flattened portion of a Lepidodendroid plant which has lost its woody axis, also the innermost and middle cortical tissues. In fig. 2, Plate XXIV. of the second of Williamson's coal plant memoirs we have a longitudinal section of *Lepidodendron selaginoides*: if we imagine all the tissues of this to be destroyed except those marked *l* and *k*, that is the epidermal or leaf-base tissues, and the tubular part of the outer bark, we have left exactly the same as those preserved in this specimen of *Lomatophloios*.

Prof. Williamson[6], in speaking of the bark tissues, remarks that near the outer surface there is a layer of prosenchymatous tissue where the cells are so elongated as to constitute a distinct "bast layer" which has exhibited a constant tendency to separate itself from the subjacent cortical tissue.

[1] *On the Organisation of the Fossil Plants of the Coal Measures*, Pt. II. Pl. XXVIII. Fig. 32 (*Phil. Trans. Royal Society*, 1872, p. 197).

[2] *Ibid.* p. 211.

[3] *Ibid.* p. 201.

[4] *On the Organisation of the Fossil Plants of the Coal Measures*, Pt. I. Pl. XXIV. Fig. 10 (*Phil. Trans.* 1871, p. 477).

[5] *On the Organisation of the Fossil Plants of the Coal Measures*, Pt. X. Pl. XX. Figs. 65, 66 (*Phil. Trans.* 1880, p. 493).

[6] *Loc. cit.* Pt. II. (*Phil. Trans.* 1872), pp. 223, 224.

Further on[1], he speaks of the very frequent occurrence, in remains of Lepidodendroid plants, of simply the epidermal layer and the semi-fibrous portion of the prosenchymatous one; all the inner tissues having been loosened by decay and eventually floated out: when the stems were prostrated, this epidermal and bast layer constituted the cylinder whose two sides were eventually brought together and flattened by superimposed pressure. In another place[2], a full description is given by the same author of a specimen which was apparently very similar to that now under consideration; the following are Prof. Williamson's words:—"I have before me at the present moment a section of a large Lepidodendron of which the woody axis and its medullary centre have disappeared, the thick cortical layer alone remaining. A large Stigmarian root has found its way into the cavity and filled it up, giving off its peculiar rootlets within the Lepidodendron cylinder. Such a specimen would inevitably mislead even a botanist whose eye was not familiar with the appearances of the two plants."

[1] *Ibid.* p. 228.

[2] *Ibid.* p. 215.

EXPLANATION OF PLATE III.

Illustrating Mr A. C. Seward's paper "Notes on *Lomatophloios macrolepidotus* (Goldg.)."

Fig. 1. Section taken at right angles to the surface of *Lomatophloios macrolepidotus*, i.e. at right angles to the leaf-bases. Actual width of section 3·5 cm.

- *a.* Position of (?) Coprolites.
- *b.* Stigmarian rootlets.
- *c.* One of the leaf-bases with space containing two Stigmarian rootlets.

Fig. 2. Part of the upper end of fig. 1 magnified. Corresponding parts shewn by the lettering in figs. 1 and 2.

Fig. 3. Outline of the specimen (*Lomatophloios macrolepidotus*).

$ab = 18$ cm.
$cd = 13$ cm.

The section shewn in fig. 1 is taken from that part indicated by **.

Fig. 4. A few of the leaf-bases ($\frac{1}{3}$ natural size), shewing leaf-scars *s*, and small indentations *i* on the swollen leaf-bases.

The whole of both surfaces of fig. 3 are covered with such leaf-bases.

(3) *On the origin of the embryos in the ovicells of Cyclostomatous Polyzoa.* By S. F. Harmer, M.A., King's College.

[*Reprinted from the Cambridge University Reporter, February* 18, 1890.]

The species investigated belonged to the genus *Crisia*, in which, as in other forms of *Cyclostomata*, the mature ovicells contain a large number of embryos. These embryos are imbedded in the meshes of a nucleated protoplasmic reticulum, which also contains a mass of indifferent cells, produced into finger-shaped processes, the free ends of which are from time to time constricted off as embryos. The embryos have, at this stage, a structure identical with that of the youngest embryos described by previous authors. After developing various organs, they escape as free larvae through the tubular aperture of the ovicell. The budding organ from which the embryos are formed makes its appearance at an early stage in the development of the ovicell. Evidence was brought forward to show that it must be regarded as an embryo, produced from an ovum. The supposed ovum is found in very young ovicells, imbedded in a compact follicle, and appears to give rise, by a remarkable process of development, to the budding organ above described. The embryos are thus produced by the repeated fission of a primary embryo developed in the ordinary way from an egg.

February 24, 1890.

Mr J. W. Clark, President, in the Chair.

The following were elected Fellows of the Society:

A. P. Laurie, M.A., Fellow of King's College.
W. H. Young, M.A., Fellow of Peterhouse.
C. Platts, M.A., Fellow of Trinity College.
A. C. Seward, M.A., St John's College.

The following Communications were made to the Society:

(1) *Vehicles used by the old Masters in Painting. Part I.* By A. P. Laurie, M.A., King's College.

I have been engaged for some time in studying the methods of painting used by the old masters, with a view to showing light on the question why modern pictures are so far from permanent. Fortunately a good deal of material exists for this enquiry, MSS. having been left containing receipts for the preparation of oils, colours and varnishes, and directions for using the same. Moreover these MSS. have been carefully edited and translated, and

books have been written on the history of painting with a view to expounding the methods used during the best period of art.

Unfortunately, however, in spite of the large amount of information thus at hand, many most important points remain obscure. It is seldom those who best know the subject who write the text-books. And these MSS. are necessarily difficult to interpret owing to the careless unscientific spirit of the time. Weights are hardly ever given and times measured by Paternosters. Names are loosely used and are often impossible to translate, and probably in most cases these receipts, collected from many sources, though useful to artists as books of reference, do not give the actual methods in use in the studio. If we imagine some one trying to reconstruct modern industrial processes from the text-books made use of in technical instruction, it would help us to realize the difficulty of reconstructing processes of the middle ages from these MSS. Unfortunately also those who have written the modern books on this subject have been artists and archæologists, but have not been chemists. Consequently whole chapters of ingenious learning may be devoted to proving that a certain process was used by the old masters, which half an hour's work in a laboratory would have shown to be impossible.

I wish here to deal with the theory developed in one of these works, namely, Eastlake's *Materials for the History of Oil Painting*, to account for the durability of the early Flemish pictures.

I shall assume that all that can be said from an historical point of view has been said by Eastlake, and that it only remains to test the truth of his conclusions by experiment.

Eastlake's book is devoted practically to expounding the method of painting used by Van Eyck and his followers.

His method is of special interest for three reasons:

He may claim to be the inventor of oil painting in the same sense that Watt invented the steam engine.

His pictures are remarkable for their durability.

He was a Flemish painter, and had consequently to deal with a damp climate very similar to our own. His methods of painting are therefore of far more value to us than the methods used in the dry climate of Italy.

In considering our subject, that is the vehicle used by the painter, we may treat it from two points of view.

Either considering the permanence of the vehicle itself, or considering the capabilities of the vehicle in protecting the pigment mixed with it from air and moisture.

I shall only consider here the question of the protection of the pigment by the vehicle. For the more we study the old methods of painting the more is the importance of this matter forced upon us.

When I began I imagined that the old masters made use of a few absolutely unalterable pigments and in this way ensured the permanence of their pictures. I soon found this view to be erroneous. Many colours were described which were so fugitive that no modern artist would use them. Red, yellow, blue, and green lakes, for instance, prepared from vegetable dyes, some of a most fugitive character under the action of air, moisture, and daylight. Many of these colours may have been only used for illuminating parchment, where, kept from light and moisture, they might doubtless be permanent; but there seems to be no reason to doubt that many were used in the oil painting of pictures. How then did these men succeed in using in oil painting colours known to be very fugitive and therefore avoided by modern artists?

In order to understand how this might be successfully done we must turn to the recent experiments of Captain Abney and Prof. Russell on the permanency of water colours, published in July, 1888 (Government report). They have tested the action of sunlight on water colours, and they find that in dry air many pigments are permanent that fade in moist air, and that in vacuo hardly any colours are altered. If then we could ensure the absence of moisture and air, many of these colours now regarded as fugitive would doubtless be permanent. Turning again to Eastlake, we find that evidently the old masters quite understood this, and that they took especial pains to lock up fugitive colours by the introduction of some varnish; but before going further let us put down briefly what vehicles were used by them for painting. They made use of a fine size prepared from parchment, gum arabic, white of egg, yolk of egg mixed with fig-tree juice and so on, but the use of oil was long difficult on account of its slowness in drying. It was soon found, however, that certain oils, such as walnut oil, poppy oil, and linseed oil, had the property of becoming converted into hard resins in time, the process technically called drying, and that this process could be hastened by exposing the oil before use for some time to the sun or by boiling it, boiled oil drying quicker than raw oil. It was next found that by boiling the oil with litharge or white lead it would dry still quicker, and was therefore more suitable as a vehicle for colours. It was also known that by dissolving in oil certain resins, such as amber, sandarac, Venice turpentine, and perhaps copal, and later mastic, that varnishes could be prepared. These facts were known before Van Eyck, and he had these materials from which to develope his method of painting in oils. There can also be no doubt that boiled oil did not afford a sufficient protection for fugitive colours, and that therefore a varnish must be ground with the colour to coat each particle, and so protect it from the action of air and moisture. So

much has, I think, been proved by Eastlake, and we have next to ask what varnish was used, or would any varnish do. On this point it is apparently impossible to get any definite historical evidence on account of the looseness with which terms were used. Nevertheless Eastlake does come to a conclusion, and states his belief that an oil varnish ground with the colour is sufficient protection whether made from amber, copal, sandarac, or Venice turpentine, and discards the tradition that Van Eyck made use of amber varnish alone. He points out that while receipts exist in the MSS. for preparing amber varnish, they also exist for the preparation of other varnishes, and there is nothing to show that amber was exclusively used. In fact it is improbable that amber varnish alone was used, as it is difficult to prepare any which is not very dark in colour. So difficult in fact is it to prepare that it can hardly be said to be an article of commerce, though occasionally varnishes claiming to be amber are sold to artists. This theory of Eastlake's is easily put to the test, and he quotes an experiment with gamboge, by Sir Joshua Reynolds, which seems to confirm it. Unfortunately, however, Prof. Church has tested this point, and finds that Eastlake's theory is not tenable. He finds that gamboge fades as quickly when mixed with copal varnish (the best of resins now in use) as it does with oil alone. We are left then with this question still to solve, "What varnish did Van Eyck use to mix with his colours?"

I have made some experiments which I think point to the correct solution of this problem. My object has been to get some rapid means of deciding whether a given vehicle was or was not permeable to moisture, and I finally hit on the following device. If we ignite sulphate of copper it loses all its water of combination, leaving a white powder which is very hygroscopic. If this powder is exposed to the air for a short time it turns green, owing to the absorption of water. I ignited therefore some sulphate of copper, and using it as a pigment ground it with boiled oil, and painted it out on three pieces of glass. One of these I placed in a desiccator, one in a warm dry room, and one in a room with the window open. In 12 hours the sulphate of copper in the damp room had turned completely green, that in the dry room slightly green, and that in the desiccator remained white. I then exposed all three in the damp room, and they were soon equally green. This showed me that I had here a delicate test of the permeability of such mediums to moisture, though this first experiment was hardly fair to the boiled oil, as it had not been allowed to harden before exposure to moisture. For no pigment used in practice is so fugitive as to be affected by exposure to moisture during the time the oil is drying, and therefore it was obviously necessary to try an experiment after the vehicles had dried completely in the

desiccator. I therefore next ground my sulphate of copper with the following vehicles, painting each out on glass:

(1) Boiled oil alone (best commercial),
(2) Copal varnish alone,
(3) Common rosin in turps,

and two with amber varnish, one prepared by dissolving amber in turps, the other by dissolving it in oil. These were placed under a desiccator and left to dry. Some of them dried very slowly, and had to be assisted by warmth till at last all were dry. I then removed the sulphuric acid from the desiccator and replaced it by water, thus leaving them in an atmosphere saturated with moisture. Very soon some of them began to turn green, and after a week all had turned green except one of those painted with amber varnish. This experiment indicates, I think, the following conclusions:

(1) That neither boiled oil nor rosin varnish, nor copal varnish, protect a pigment absolutely from moisture.

(2) That amber varnish properly prepared does do so. The amber varnish that protected the sulphate of copper was that prepared with turps. The one prepared with oil failed to do so. Therefore we may take it that Van Eyck probably used an amber varnish.

(2) *On the Action of the Copper Zinc Couple on dilute solutions of Nitrates and Nitrites* (NaHO *and* KHO *being absent*). By James Monckman, D.Sc. Lond., Downing College, Cambridge.

It is usually stated that when a dilute solution of a nitrate, containing a copper zinc couple, is boiled for some time, the whole of the nitrogen in the nitrate is given off as ammonia, which may be used as a means of estimating the quantity of nitric acid. When NaHO or KHO is added to the liquid good results are obtained, but when no such addition is made the quantities of ammonia found, by students working in the University Laboratory, has fallen so far short of the true amount, that it appeared to point to some other reaction taking place at the same time. I was therefore asked to examine it carefully in order to discover why the ammonia found was not equivalent to the nitrate used and what became of the remainder.

Quantities obtained from KNO_3. My first endeavour was to try what kind of results could be obtained by the method, how far the various experiments could be made to agree in the quantity of ammonia evolved and the amount of nitrogen that disappeared. A very large number of experiments were per-

formed in order if possible to get some kind of regularity in the numbers found, but without success.

The method employed, is that one described in text-books on quantitative analysis. It consists in placing a quantity of well washed CuZn couple in a dilute solution of the salt (KNO_3) and boiling. The steam is conducted through a condenser into a measured quantity of standard acid, so arranged that all the ammonia is absorbed.

It is usually stated that the reaction is at an end in one hour, and that afterwards no further evolution of ammonia takes place. As this did not appear to be the case in the first set of observations, the boiling was continued in the succeeding ones until no more ammonia was given off. In doing this it was found necessary to evaporate to dryness, then add more water and evaporate again. In some cases this process was repeated six times before every trace of the alkaline gas was removed.

Sometimes the reaction stopped long before the whole of the water passed over, and yet on the addition of more water a further evolution of gas took place. This was caused by the oxide of zinc, produced during the progress of the reduction, forming a covering to the copper zinc couple and thus preventing the access of the liquid. When more water was added, some of this oxide was washed off and the surface thus exposed continued the reduction of the nitrate.

The experiments gave numbers varying from 29 per cent. of the calculated quantity up to 67 per cent. when the ammonia had been produced slowly, while it rose from 70 to 80 per cent. when the boiling was urged.

Careful search was made for insoluble or soluble nitrogen compounds, which might resist the action of the hydrogen, such as basic nitrates, but no such bodies could be detected.

Nitrites were however found to be produced in all cases. I give the numbers for these experiments as an example.

On the 1st boiling	*A* gave	55	per cent.,	*B* gave	50,	*C* gave	55
2nd	„	3	„	„	20	„	8
3rd	„	10	„	„	0	„	9
4th	„	7	„	„	—	„	6
5th	„	0	„	„	—	„	0
		75			70		78

Thus the highest number was a little more than 20 per cent. short of the calculated quantity. As the highest numbers were obtained from those solutions which had been made to boil most strongly, experiments were designed to try if the variation of the temperature caused any alteration in the chemical reaction.

Effect of temperature. It was found that when the liquid was evaporated by heating over a water-bath no ammonia was produced, while the whole of the nitrate was decomposed, nitrite being the only substance left in the liquid.

Action on KNO_2. When a solution of KNO_2 was used instead of KNO_3 it was found that after boiling some time it was completely decomposed and the nitrous acid disappeared from the solution without giving any ammonia.

As no nitrogen compounds could be found in the liquid and no ammonia was given off nor yet any acid compound of nitrogen*, I was driven to the conclusion that it came away as either N or N_2O, probably the former, and that probably the ammonia acted upon the nitrous acid producing nitrite of ammonia which was again broken up into N and H_2O.

During violent ebullition excess of ammonia was produced and driven off, but during gentle evaporation it united with the nitrous acid, which was formed in sufficient quantity to combine with it.

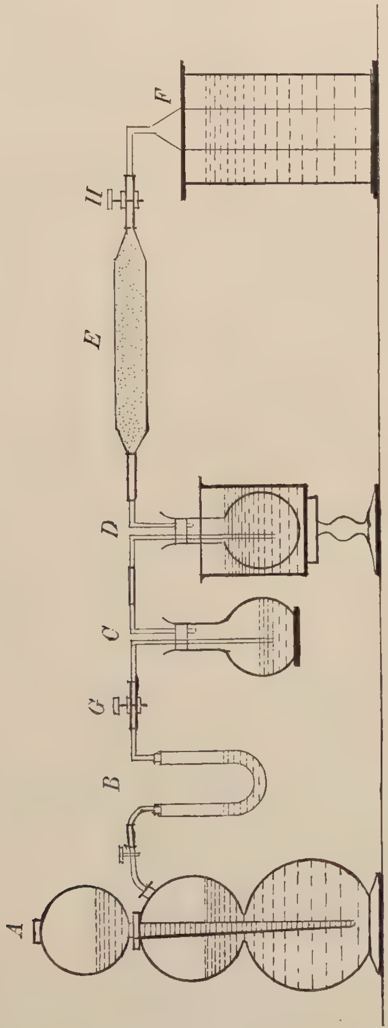
This appears more probable because, in the first experiments with nitrates, the nitrate was produced by the reducing action of the hydrogen, at the same time as the ammonia, while in the latter case (the nitrites) there was excess of this body from the very beginning of the reaction. In the first ammonia escaped, in the second, not.

Methods of testing for N. In the experiments described the presence or absence of hydrogen gas was of no importance, but when it becomes a question of testing for nitrogen and estimating its volume, a large quantity of hydrogen becomes inconvenient. I first tried to avoid producing the gas, by using a solution of the nitrite of sufficient strength to absorb the whole of it. I found that it did not work well, the action being very slow and little gas coming away, and as it could not be more than a qualitative method it was abandoned.

Next it was proposed to sweep out the air from the tubes by a current of gas (CO_2) and after boiling the solution to carry the gas produced into a receiver in the same manner. The CO_2 was to be absorbed in the usual way. The danger of producing a compound of NH_3 and CO_2 which might, in presence of the nitrite of potash, produce nitrogen after the manner of AmCl and KNO_2, formed an obstacle to its use. I was therefore compelled to substitute H.

* Twice I found a very slight acid reaction produced by the gas evolved and twice considerable quantities of NO were evolved, but after washing the couple until it was perfectly free from acid or acid salt, the dilute solution ceased to give NO.

The method of working will be understood by the aid of the diagram.



A is an apparatus for generating H which passes through a solution of silver nitrate in *B*, the nitrate of potash and the CuZn couple are placed in *C*, while *D* contains the acid for absorbing the ammonia. One vessel only is used to keep the

capacity of the whole as small as possible, the current of steam was kept very gentle and the water in the outside vessel quite cold, *E* contains the granulated oxide of copper and *F* the receiver.

After passing the gas through *B, C, D, E* for 18 hours it was joined up to *F*, and sufficient gas sent through to force the water in the receiver down about three inches below the surface of the water in the outer vessel. It was then stopped and the clip *G* closed to prevent steam passing back into *B*.

The liquid in *C* was then boiled gently for three hours, the quantity of gas in the receiver was prevented from becoming too great by carefully heating the oxide in *E*. Finally by a current of hydrogen from *A* the whole of the gas was swept out of the tubes; as before most of this was absorbed by heating the oxide in *E*, so that a considerable quantity was passed through the tubes *B, C, D*. The clip *H* was next closed and the quantity of N in the receiver determined.

In order to decompose any nitrate or nitrite that might remain in *C*, HKO was added to the solution and the whole boiled until all the salt was decomposed, after which the ammonia was determined.

The results appear in the following table:

The weight of KNO_3 in the solution was ·225 grms.

of which the N would weigh ·03118 grms.

The result of the previously described experiment was

N evolved as gas	·02304	grms.
N evolved as NH_3	·009	,,
Total	·03204	
Too much by	·00086	grms.

Given in per cent. of the salt used:

calculated,	N is	13·861	per cent.,
found,	N is	14·24	,, ,,
excess,		·379	,, ,,

The only other salt tried by me was the nitrate of ammonia, which gave nitrite on gently boiling with the couple. As that salt is decomposed into N and H_2O on boiling I did not consider it necessary to prove the evolution of N in this case.

In conclusion I wish to say that no equation has been given, because the proportion of NH_3 evolved so evidently depended upon the temperature, or rate of boiling, that it would be misleading; the reaction may be represented however by the two

following, combining them in varying quantities according to the temperature:

$$\begin{array}{lll} (\alpha) & KNO_3 + H_8 & = KHO + 2H_2O + NH_3, \\ (\beta) & KNO_3 + H_2 & = KNO_2 + H_2O, \\ (\alpha+\beta) & KNO_2 + H_2O + NH_3 & = KHO + NH_4NO_2 \\ & NH_4NO_2 & = N_2 + 2H_2O. \end{array}$$

During the course of this research I received much valuable advice from Mr Sell and Mr Fenton of the University Laboratory for which I wish to acknowledge my obligation.

In the previous description I have mentioned the fact that the strong solutions of nitrates and nitrites did not produce the same reactions, failing to produce a sufficient quantity of gas.

Probably this arose from the same kind of difference being produced in the reaction by the variation of the density of the solution, as by the temperature at which it took place. This view was not put to any test.

(3) *On certain Points specially related to Families of Curves.* By J. BRILL, M.A., St John's College.

1. The following communication is intended as a sequel to my paper "On the Geometrical Interpretation of the Singular Points of an Equipotential System of Curves*," and in it I propose to show that the theorems which I established with regard to Equipotential Systems are, with certain qualifications, true of a much more extensive class of systems. It is to be understood that the remarks contained in this paper refer to algebraical systems, as it is impossible to reason in a general manner about transcendental curves where imaginary geometry is concerned.

In my paper entitled "Orthogonal Systems of Curves and of Surfaces†" I showed that if ξ and η be the parameters of two families of curves constituting an orthogonal system, and if h_1 and h_2 be defined by the equations

$$\left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\xi}{\partial y}\right)^2 = h_1^2$$

and
$$\left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial y}\right)^2 = h_2^2,$$

then the value of the expression

$$\frac{h_2 d\xi + ih_1 d\eta}{dx + idy}$$

* *Proc. C. P. S.*, VI., pp. 313—320.
† *Proc. C. P. S.*, VI., pp. 230—245.

is independent of the ratio $dy : dx$. Now it is evident that there must be certain points or loci for which the inverse of this expression vanishes, and therefore the expression itself becomes infinite. These points or loci may be considered as a generalization of the singular points of equipotential systems.

I do not propose to discuss the general case, but to confine myself to a large and important series of cases in which the ratio of the two h's is some definite function of x and y, which function of x and y is the same for all systems of a particular class. I shall write $h_2/h_1 = \phi(x, y)$, and instead of the expression given above shall consider the expression

$$\frac{\phi(x, y)\, d\xi + i d\eta}{dx + i dy}.$$

The fact that this expression has a value independent of the ratio $dy : dx$, requires the existence of the relations

$$\phi(x, y)\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y} \text{ and } \phi(x, y)\frac{\partial \xi}{\partial y} = -\frac{\partial \eta}{\partial x},$$

and the value of the expression may be written in either of the forms

$$\phi(x, y)\left\{\frac{\partial \xi}{\partial x} - i\frac{\partial \xi}{\partial y}\right\} \text{ or } \frac{\partial \eta}{\partial y} + i\frac{\partial \eta}{\partial x}.$$

Moreover it follows that the value of the expression

$$\frac{dx + i dy}{\phi(x, y)\, d\xi + i d\eta}$$

is independent of the value of the ratio $d\xi : d\eta$, and this requires the existence of the relations

$$\frac{\partial x}{\partial \xi} = \phi(x, y)\frac{\partial y}{\partial \eta} \text{ and } \frac{\partial y}{\partial \xi} = -\phi(x, y)\frac{\partial x}{\partial \eta},$$

and its value may be written in either of the forms

$$\frac{\partial y}{\partial \eta} - i\frac{\partial x}{\partial \eta} \text{ or } \frac{1}{\phi(x, y)}\left\{\frac{\partial x}{\partial \xi} + i\frac{\partial y}{\partial \xi}\right\}.$$

It is also to be remarked that the relations given above lead to the equation

$$\frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} = 0.$$

Thus, if ds be the length of an elementary arc, we have

$$ds^2 = dx^2 + dy^2 = \left\{\frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta\right\}^2 + \left\{\frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta\right\}^2$$

$$= \left\{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2\right\} d\xi^2 + \left\{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2\right\} d\eta^2.$$

2. We proceed to discuss the properties of the loci of ultimate intersections of the two families of the orthogonal system. Suppose that P is the point (x, y), and that Q is a neighbouring point on the curve of the family η which passes through P; then by the preceding article we have

$$PQ^2 = \left\{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2\right\} d\xi^2.$$

If this vanish independently of the value of $d\xi$, then two consecutive curves of the family ξ cut at the point P. The vanishing of the said expression requires that either

$$\frac{\partial x}{\partial \xi} = \frac{\partial y}{\partial \xi} = 0,$$

or

$$\frac{\partial x}{\partial \xi} \pm i \frac{\partial y}{\partial \xi} = 0.$$

The first of these conditions has reference to the existence of a real locus of ultimate intersections, as is otherwise evident. The equations

$$\frac{\partial x}{\partial \xi} = 0 \quad \text{and} \quad \frac{\partial y}{\partial \xi} = 0,$$

if expressed in terms of x and y, would denote two curves which would in general intersect in a set of discrete points. If however the expressions $\partial x/\partial \xi$ and $\partial y/\partial \xi$ had a common factor, then two branches, one belonging to each curve, would coincide, and consequently the curves of the family ξ would have a locus of ultimate intersections not consisting wholly of discrete points. We can however show that there are restrictions on the form of such a factor should it exist; for if this factor be not such that if equated to zero it secures that $\phi(x, y)$ shall at the same time vanish, then it follows that at all points for which

$$\frac{\partial x}{\partial \xi} = 0 \quad \text{and} \quad \frac{\partial y}{\partial \xi} = 0.$$

We have also

$$\frac{\partial x}{\partial \eta} = 0 \quad \text{and} \quad \frac{\partial y}{\partial \eta} = 0.$$

Thus at all points of the locus of ultimate intersections we have

$$dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta = 0$$

and $$dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta = 0.$$

These equations give $x = \text{const.}$ and $y = \text{const.}$, and thus the locus of ultimate intersections consists of a set of discrete points. Moreover this set of points is such that the coordinates of each of them make the expression with which we started infinite.

Thus we have, in general, that if two consecutive members of one of the two families intersect, then all their points of intersection are points of the character under discussion, and it follows that all the members of the family pass through this set of points. The only exception to this is that the curves of the ξ family may possibly have for a locus of ultimate intersections the locus of the points for which $\phi(x, y)$ vanishes, and the curves of the η family may possibly have for a locus of ultimate intersections the locus of the points for which the same expression becomes infinite.

The question now arises whether these special loci, should they exist, belong to the generalization of the singular points of an equipotential system. This depends on how we define that generalization. If we define it with the aid of our original expression, we see that these loci do not belong to it if they arise from the vanishing of h_1 or h_2, but if they take their origin from either of these quantities becoming infinite then they do so belong. If on the other hand we define it with the aid of the expression

$$\frac{\dfrac{d\xi}{h_1} + i\dfrac{d\eta}{h_2}}{dx + i\,dy},$$

then the exact contrary is the case.

It does not follow that either of the two families should be such that all its members pass through the points under discussion, although it is true that if any two belonging to one family do so, all belonging to that family will. Also it is evident, on account of the orthogonal property, that if one of the families be such that all its members pass through the said system of points, then those of the other family do not do so, only one member of that family passing through any one of the points. It is however conceivable that there may be cases in which the members of one family pass through some of the points, and the members of the other family through others of them.

3. We now pass on to discuss the case of imaginary loci of ultimate intersections. These are given by the condition

$$\frac{\partial x}{\partial \xi} \pm i \frac{\partial y}{\partial \xi} = 0.$$

It will only be necessary to consider one of these two cases, as the discussion of the two will be exactly similar, and we will choose the one given by the upper sign. The relation

$$\frac{\partial x}{\partial \xi} + i \frac{\partial y}{\partial \xi} = 0$$

requires also the existence of the relation

$$\frac{\partial y}{\partial \eta} - i \frac{\partial x}{\partial \eta} = 0$$

i.e.

$$\frac{\partial x}{\partial \eta} + i \frac{\partial y}{\partial \eta} = 0,$$

unless the first of these relations makes $\phi(x, y)$ zero or infinite. Thus, with this reservation, we have at all points of an imaginary locus of ultimate intersections

$$dx + idy = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta + i \left(\frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta \right) = 0,$$

i.e.

$$x + iy = \text{const.}$$

If we had taken the lower sign we should have had

$$x - iy = \text{const.}$$

Thus we see that if there be any imaginary locus such that the coordinates of all points on it make $\phi(x, y)$ zero or infinite, then it is conceivable that this may be part of the locus of ultimate intersections of one of the families of the system. If not, then the imaginary loci of ultimate intersections, whensoever they exist, are collections of straight lines passing through the circular points at infinity. Each of these imaginary straight lines will pass through one real point, and these points will belong to the set of points characterized above. Further, as we are only considering algebraical curves, for any imaginary point of intersection $(a + ib, c + id)$ there will be a conjugate point of intersection $(a - ib, c - id)$. Thus it is evident from the reasoning contained in my former paper that the imaginary straight lines occur in pairs, the two members of each pair being conjugate and intersecting in a real point, which is one of the points in question.

The reasoning of this article proves only that if imaginary loci of ultimate intersection exist, then with the restriction speci-

fied above they are of the character described. Every locus of the form $x \pm iy = \text{const.}$ will cause the expression for ds to vanish, but it is only such of these loci that pass through the points under discussion that form part of the locus of ultimate intersection of the members of either of the families.

4. As the demonstration of the preceding article is attended with some difficulties, it will be perhaps well to give another demonstration modelled on one given by Kummer in the paper referred to in the communication to which this is a sequel. If we write

$$\xi + \eta = 2u \text{ and } \xi\eta = v^2,$$

then we may consider the parameters of our two families of curves as the roots of the equation

$$\alpha^2 - 2\alpha u + v^2 = 0.$$

To find the loci of ultimate intersections of the curves given by this equation, we have

$$\alpha - u = 0;$$

and the loci we are in quest of are given by

$$u^2 = v^2,$$

or $$u = \pm v.$$

We will now discuss the direction of the tangent of one of these loci. We have

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = \pm\left\{\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx}\right\},$$

$$\therefore \frac{dy}{dx} = -\frac{\dfrac{\partial u}{\partial x} \pm \dfrac{\partial v}{\partial x}}{\dfrac{\partial u}{\partial y} \pm \dfrac{\partial v}{\partial y}}.$$

It remains to express the condition of orthogonality of the two families of curves. We have the equations

$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial x} = 2\frac{\partial u}{\partial x},$$

$$\eta\frac{\partial \xi}{\partial x} + \xi\frac{\partial \eta}{\partial x} = 2v\frac{\partial v}{\partial x};$$

and from these it follows that

$$(\xi - \eta)\frac{\partial \xi}{\partial x} = 2\left\{\xi\frac{\partial u}{\partial x} - v\frac{\partial v}{\partial x}\right\}, \quad (\xi - \eta)\frac{\partial \eta}{\partial x} = -2\left\{\eta\frac{\partial u}{\partial x} - v\frac{\partial v}{\partial x}\right\}.$$

Similarly we should obtain

$$(\xi-\eta)\frac{\partial\xi}{\partial y}=2\left\{\xi\frac{\partial u}{\partial y}-v\frac{\partial v}{\partial y}\right\},\quad (\xi-\eta)\frac{\partial\eta}{\partial y}=-2\left\{\eta\frac{\partial u}{\partial y}-v\frac{\partial v}{\partial y}\right\}.$$

Substituting these values in the equation

$$\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}=0,$$

we have

$$\left(\xi\frac{\partial u}{\partial x}-v\frac{\partial v}{\partial x}\right)\left(\eta\frac{\partial u}{\partial x}-v\frac{\partial v}{\partial x}\right)+\left(\xi\frac{\partial u}{\partial y}-v\frac{\partial v}{\partial y}\right)\left(\eta\frac{\partial u}{\partial y}-v\frac{\partial v}{\partial y}\right)=0,$$

or

$$v^2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right\}-2uv\left\{\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right\}=0.$$

Now if $u=\pm v$, this equation becomes

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\pm 2\left\{\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right\}=0,$$

i.e.

$$\left\{\frac{\partial u}{\partial x}\pm\frac{\partial v}{\partial x}\right\}^2+\left\{\frac{\partial u}{\partial y}\pm\frac{\partial v}{\partial y}\right\}^2=0,$$

or

$$\frac{\dfrac{\partial u}{\partial x}\pm\dfrac{\partial v}{\partial x}}{\dfrac{\partial u}{\partial y}\pm\dfrac{\partial v}{\partial y}}=\pm i.$$

This proves our point. This proof is not identical with Kummer's, though modelled on it*. The mistake made by Kummer was to assume that the locus of ultimate intersections necessarily constituted a proper envelope, and thus that the points under discussion were necessarily foci. I have however dwelt at sufficient length upon this point in my former paper, and consequently it will need no further discussion here.

5. It now only remains to point out a few examples of classes of systems that come under the case we have been discussing. Our first example will be drawn from systems of curves which furnish solutions of the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=c^2u.$$

* If we express the equation $u^2=v^2$ in terms of ξ and η it becomes $\xi^2+\eta^2+\xi\eta=0$, and the loci of ultimate intersections would therefore appear to be wholly imaginary (exception being made of the points spoken of above). Kummer's method of investigation therefore gives rise to the question whether, if a family of algebraical curves have an envelope which is not a set of discrete points, their orthogonal trajectories will necessarily consist of a family of transcendental curves.

Suppose that we have two functions u and v, which are connected by the relations

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = cu \text{ and } \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -cv.$$

From these we easily deduce

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) = c(u - iv),$$

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u - iv) = c(u + iv).$$

Thus we have

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\}(u + iv) = c^2(u + iv),$$

and therefore u and v are both solutions of the equation given above.

Further, the relations connecting u and v may be written in the form

$$\frac{\partial}{\partial x}(e^{-cx}u) = \frac{\partial}{\partial y}(e^{-cx}v) \text{ and } \frac{\partial}{\partial y}(e^{cx}u) = -\frac{\partial}{\partial x}(e^{cx}v).$$

These relations show that we may express u and v in terms of two new functions ϕ and ψ as follows:

$$u = e^{cx}\frac{\partial \phi}{\partial y},\ v = e^{cx}\frac{\partial \phi}{\partial x},\quad u = -e^{-cx}\frac{\partial \psi}{\partial x},\ v = e^{-cx}\frac{\partial \psi}{\partial y}.$$

From this it follows that

$$e^{2cx}\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and } e^{2cx}\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

Another good example is given by cases of irrotational fluid motion symmetrical with respect to an axis. In this case we have a velocity potential ϕ and a stream function ψ connected by the relations

$$r\frac{\partial \phi}{\partial r} = \frac{\partial \psi}{\partial z},\quad r\frac{\partial \phi}{\partial z} = -\frac{\partial \psi}{\partial r}.$$

These two examples will suffice to show that among the classes of systems discussed in this paper there are some which have applications to problems of interest.

March 10, 1890.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:

(1) *On the germination of* Acacia sphaerocephala. By W. GARDINER, M.A., Clare College.

[*Reprinted from the Cambridge University Reporter, March* 18, 1890.]

Seeds of this well-known myrmecophilous plant have been lately germinated, both at Kew and Cambridge, affording opportunity of observing seedlings in all stages of growth. The striking manner in which this plant exhibits definite structural adaptations for the benefit of the ant colonies, which so efficiently garrison it, has suggested that the structures in question possibly owe their formation to the action of the ants themselves, and arise in consequence of local stimulation produced by biting (when in search of sweet sap) or even by stinging. Actual observation shows that as the seedlings assume the adult foliage, the stipular thorns, the petiolar glands, and the "food bodies" all develope quite normally, and in spite of the ants being absent. It is clear therefore that in plants which exist at the present time the structures appear without the intervention of ants. It may, however, be urged that they were in the first instance brought into existence by ants, through the stimulation of ancestral forms, such stimulation having been persistent and extending over many generations. A study of germinating seedlings does not appear to support this view. The development of the several structures can be traced so gradually and through so many transition forms that there appear to be strong grounds for believing that the whole of the complicated arrangements and of the several organs concerned are the outcome of variation, and that the ants have done little more than take advantage of the results of such variation. Moreover no absolutely new organs are present, the thorns being as well, or even better, developed in other species of *Acacia; e.g. Acacia latronum:* the petiolar nectaries being well known and common structures: and the food bodies (as their development shows) being hypertrophied "Reinke's glands."

(2) *Additional note on the thickening of the stem in the* Cucurbitaceæ. By M. C. POTTER, M.A., St Peter's College.

Hitherto very few Dicotyledons have been described which possess a ring of normal collateral bundles and whose stem at the same time does not increase in thickness by means of a cambial ring. De Bary[1] mentions under this category the *Saurureæ*

[1] *Comp. Anat.* Eng. Ed. p. 454.

and some species of *Ranunculus;* and to this short list we may add some water plants, e.g. *Hippuris* and *Myriophyllum,* etc.

The reason why so few Dicotyledons do not increase in thickness may be sought for in the habit and mode of life of this important series of the vegetable kingdom. Many Dicotyledons are perennial, being trees or shrubs, and require, as they grow older and larger, that additional phloem and xylem should be continually formed both for purposes of nutrition and also for mechanical support; and this is accomplished by means of the cambium ring which adds continually new phloem and xylem to the vascular bundles. Among the herbaceous Dicotyledons increase in thickness is also the rule; for these plants having annual stems require that sufficient phloem and xylem should be made for purposes of nutrition before the bundles are closed, and that the mechanically supporting tissue should be sufficiently strengthened to bear the increasing strains as the plant grows larger; and hence a cambium ring is necessary for nutrition and mechanical support.

The few Dicotyledons mentioned above live under special conditions;—*Hippuris, Myriophyllum,* being water plants, require little xylem or supporting tissue; and hence continual additions to the xylem are not needed, and so the bundles are closed. Again the *Saurureæ* and the species of the *Ranunculaceæ* (viz. *Caltha palustris* and species of *Ranunculus*) are marsh plants, which do not attain to any great size and whose habit is very similar to that of water plants; and hence these do not require the continual addition of xylem and phloem to their vascular bundles.

The *Cucurbitaceæ* may roughly be divided into (1) herbaceous climbers with annual stems and (2) woody perennial climbers. The former on the one hand have the stem strengthened by a ring of sclerenchymatous tissue situated in the cortical tissue between the epidermis and the vascular bundles. The support derived from this sclerenchyma and the xylem which is formed before the bundles are closed, together with that which is derived from the external object upon which the plant climbs are sufficient; and hence additional xylem is not required for purposes of support, the plant only requiring that the amount necessary for nutritive purposes should be made before the bundle is closed. The latter on the other hand have no ring of sclerenchyma, and derive their support from the xylem and external objects; but since these plants are perennial, and continually make fresh leaves and branches, they require the constant addition of new xylem and phloem to their stems; and from this fact we have the reason why a cambium ring is present.

The explanation why an additional layer of phloem on the inside of the xylem is needed by the members of this order must again be sought for in the special conditions under which they

live. We have seen that in the herbaceous species a large amount of xylem is not needed; but, by comparison with other herbaceous plants, we see that a considerable amount of phloem is necessary. This phloem receives considerable additions before the bundles are closed, and we may infer that it is more beneficial that the phloem should be divided by the xylem than that it should be placed only on the exterior of the xylem.

In the woody perennial species, the amount of internal phloem is not large; and the requisite amount is formed in the normal way by the cambium as growth in thickness takes place. These stems have no sclerenchymatous ring, and hence derive their support from the xylem and external objects, and hence, as the tree grows larger, continual additions are made to the vascular bundles.

[*Received March* 9, 1890.]

(3) *Note on the action of Rennin and Fibrin-ferment.* By A. S. Lea, Sc.D., Caius College, and W. L. Dickinson, Caius College.

[*Reprinted from the Cambridge University Reporter, March* 18, 1890.]

The experiments which were demonstrated to the Society were made in order to verify some recent statements of Fick (*Pflüger's Arch.*, Bd. 45, 1889, S. 293). This observer had urged that rennin certainly, and fibrin-ferment probably, produced their respective actions on milk and blood-plasma without the necessary contact, throughout the whole fluid mass, of the ferment molecules with the molecules of casein and fibrinogen. If this were so, then the mode of action of these ferments would be strikingly different from that of the ordinary digestive ferments. Fick based his views upon the result of an experiment made by placing a glycerine extract of rennin at the bottom of a tube, carefully pouring some milk on the top of the glycerine and observing that, without mixing the two fluids, the milk rapidly clotted throughout its entire mass.

The authors experimented by warming milk to 40° C. in a test-tube, and then carefully introducing, by means of a fine glass tube, an active extract of rennin below the milk. The constant result of this experiment was that a clot was rapidly formed at the junction of the two fluids, but that not until the lapse of several hours was the milk clotted throughout. When a similar experiment was made with dilute salt-plasma a narrow clot of fibrin was similarly formed at the junction of the plasma and fibrin-ferment solution. But unlike the case with milk, the super-

jacent plasma was never clotted up to its surface even after 24 hours' digestion at 40°. When however at the end of this time the ferment and the plasma were *mixed* by shaking the tube, clotting throughout the whole mass speedily occurred. From these experiments the authors concluded that Fick's views are not tenable, and that there is no reason for supposing that the mode of action of these ferments differs essentially from that of other ferments. They explained the results obtained by Fick and themselves in the case of milk as due to the inevitable mixing of traces of the rennin with the milk, and pointed out how slight an amount of mixing would suffice to produce the observed result by calling attention to the fact that rennin will clot 400,000—800,000 times its own weight of casein. Their results obtained with dilute salt-plasma were even more striking, since with this the clot never extended, even after prolonged digestion, more than a few millimeters above the junction of the surfaces of the ferment solution and plasma.

(4) *On some Skulls of Egyptian Mummied Cats.* By W. BATESON, M.A., St. John's College.

[*Reprinted from the Cambridge University Reporter, March* 18, 1890.]

Six skulls and two restored heads of Egyptian mummy-cats were shown in illustration of the early history of the domestication of the cat. The specimens indicate that the cats embalmed by the Egyptians were of at least two kinds, and that the larger variety was of much greater size than that usually reached by either the modern domestic cat or the wild cat of Europe. These facts have been already pointed out by de Blainville and Nehring, but on comparison with a series of modern skulls it is not possible to support the attempt to refer these animals to any particular species of cat. The presumption is rather that cats of many kinds and sizes, possibly distinct, and probably including *Felis serval* and *F. caligata* (? = *F. maniculata* and *F. caffra*), were all thus embalmed; but whether these animals were all domesticated or whether some were merely collected from time to time there is no evidence to show.

Pupa-cases of the maggots which had lived in these heads were also exhibited.

April 28, 1890.

Mr J. W. CLARK, PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:

(1) *On the series in which the exponents of the powers are the pentagonal numbers.* By Dr GLAISHER.

The principal result was that the square of the series

$$1 + q + q^2 + q^5 + q^7 + q^{12} + q^{15} + q^{22} + q^{26} + \&c.$$

(which is known to be equal to the product

$$(1 + q)(1 + q^2)(1 - q^3)(1 + q^4)(1 + q^5)(1 - q^6)(1 + q^7)\ldots)$$

is the series

$$1 + E(13)\,q + E(25)\,q^2 + E(37)\,q^3 + E(49)\,q^4 + \&c.$$

where $E(n)$ denotes the excess of the number of divisors of n which $\equiv 1$, mod 4, over the number which $\equiv 3$, mod 4.

(2) *The Influence of Electrification on Ripples.* By J. LARMOR, M.A., St John's College.

The relation between the period and the wave length of ripples on the surface of a liquid must obviously be sensibly affected by an electric charge communicated to the surface.

To investigate the amount of the effect, let us take the origin of coordinates on the surface, the axis of y vertically downwards, and the axis of x along the direction of propagation of the ripples.

A suitable form for the electric potential V above the liquid is

$$V = -Ay + ACe^{-my}\cos mx.$$

The equation of the surface ($V = 0$) is then

$$y = C\cos mx,$$

and the surface density σ of the electric charge is equal to $A/4\pi$.

The velocity potential of the wave motion is a function ϕ which satisfies

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0,$$

and gives at the surface ($y = 0$),

$$\frac{\partial\phi}{dy} = \frac{dC}{dt}\cos mx.$$

Thus when the depth is so great that the ripples do not disturb the bottom

$$\phi = -\frac{1}{m}\frac{dC}{dt}e^{-my}\cos mx,$$

as this leads to zero velocity at a great depth.

The electric charge diminishes the surface pressure by $-\sigma\frac{\partial V}{\partial n}$, or $\frac{1}{4\pi}\left(\frac{\partial V}{\partial n}\right)^2$, or approximately $\frac{1}{4\pi}\left(\frac{\partial V}{\partial y}\right)^2$, which is equal to

$$\frac{A^2}{4\pi}(1 + 2Cm\cos mx).$$

The surface tension T diminishes the surface pressure by

$$T\frac{\partial^2 y}{\partial x^2}, \text{ or } -TCm^2\cos mx.$$

Now the equation of fluid pressure is

$$\frac{p}{\rho} = \text{const.} + gy - \frac{\partial\phi}{\partial t} - \tfrac{1}{2}v^2;$$

so that we must have at the surface, approximately,

$$-\frac{A^2}{4\pi\rho}(1 + 2Cm\cos mx) + \frac{TCm^2}{\rho}\cos mx$$
$$= \text{const.} + gC\cos mx - \frac{1}{m}\frac{d^2C}{dt^2}\cos mx,$$

which requires

$$\frac{d^2C}{dt^2} + C\frac{m^2}{\rho}\left(Tm + \frac{g\rho}{m} - \frac{A^2}{2\pi}\right) = 0.$$

This gives the periodic time

$$\tau = \frac{2\pi}{m}\rho^{\frac{1}{2}} \bigg/ \left(Tm + \frac{g\rho}{m} - \frac{A^2}{2\pi}\right)^{\frac{1}{2}}$$
$$= \lambda \bigg/ \left(\frac{2\pi T}{\lambda\rho} + \frac{g\lambda}{2\pi} - \frac{8\pi\sigma^2}{\rho}\right)^{\frac{1}{2}},$$

where λ is the wave length; and the velocity of propagation is

$$\left(\frac{2\pi T}{\lambda\rho} + \frac{g\lambda}{2\pi} - \frac{8\pi\sigma^2}{\rho}\right)^{\frac{1}{2}}.$$

The result is in fact the *same as would be produced by a decrease in surface tension of amount*

$8\pi\sigma^2/m$, or $4\sigma^2\lambda$.

This quantity depends on λ, as might have been expected, for the mechanical effects of an electric charge on the surface cannot be represented as a diminution of surface tension. To produce a simple reduction of tension electrically we must have the double condenser larger than has been assigned by von Helmholtz as the cause of voltaic polarization.

When, as in the case of voltaic polarization, the ripples occur at the interface between two liquids of densities ρ_1 and ρ_2 the above formulæ will clearly be applicable on substitution of $\rho_1+\rho_2$ for ρ and $g(\rho_1-\rho_2)/(\rho_1+\rho_2)$ for g.

The actual values here deduced come from the form of ϕ that belongs to fluid of some depth compared with λ; but it is obvious that the surface tension effect combines with the electric effect in the same way in every case, and that the statement just made holds generally.

As the length of the ripples diminishes, the effect of the electrification is ultimately negligible compared with that of the surface tension, though it persists much longer than the influence of gravity.

In the special case considered above the period becomes imaginary if λ lie between the values

$$\frac{8\pi^2\sigma^2}{g\rho}\left(1 \pm \sqrt{1-\frac{T}{2\sigma^2}\Big/\frac{8\pi^2\sigma^2}{g\rho}}\right)^2;$$

so that, if σ can be made so great that these limits are real, the wave lengths that lie between them cannot exist. For a given period there will be a wave length above these limits for which gravity is chiefly operative, and one below them for which surface tension is chiefly operative.

To obtain a rough numerical estimate: On a circular plate of radius a changed to potential V the electric density at distance r from the centre is $\sigma = V/\pi^2(a^2-r^2)^{\frac{1}{2}}$, while on a sphere of the same radius the electric density is $\sigma = V/4\pi a$, which is rather less than that at the centre of the plate. At the centre of the plate the effective diminution of surface tension will be $4V^2\lambda/\pi^4a^2$. If we take a 10 cm. and λ 1 cm. this gives about $V^2/2500$ in C.G.S. units. The value 33 for V makes the striking distance in air between balls 2 cm. in diameter about ·3 cm., while according to Mascart the value 400 makes the striking distance between balls 2·2 cm. in diameter about 10 cm.; the former value gives an effective diminution of surface tension of $\frac{11}{25}$, the latter gives 64. For water the actual surface tension is about 80, for mercury 540. The electric effect is therefore considerable: thus Prof. C. Michie Smith (*Proc. R. S. Edin.*, March, 1890) has observed an effective diminution of 20 per cent. in the case of mercury owing to electrification.

(3) *On Sir William Thomson's estimate of the Rigidity of the Earth.* By A. E. H. LOVE, M.A., St John's College.

(*Abstract.*)

In Thomson and Tait's *Natural Philosophy* (part II. art. 834 sq.) there is given an estimate of the rigidity of the earth derived from a consideration of tidal phenomena. For the purpose of obtaining such an estimate the earth is regarded as a homogeneous elastic solid sphere, of gravitating matter and incompressible, which is supposed to be strained by its own gravitation and by the action of external disturbing bodies such as the moon. The amount of the tidal distortion is measured by the ratio of the spherical harmonic deviation of the disturbed surface from the mean spherical figure to the radius of the mean sphere, and this is compared with the corresponding ratio in case the rigidity is annulled, i.e. in the case of the earth regarded as a homogeneous fluid sphere disturbed by the same system of forces. We may for shortness speak of the first dynamical system as a 'solid earth,' and of the second as a 'fluid earth.' It is shown that the ratio of the amount of tidal distortion in a 'solid earth' of the same rigidity as glass to that in a 'fluid earth' is about ·612 or nearly $\frac{3}{5}$, while if the rigidity be taken to be that of steel the ratio is about ·321 or nearly $\frac{1}{3}$. In a perfectly rigid 'solid earth' the ratio would be zero. The height of the tide as given by the observable rise and fall of the sea relatively to the land is the difference of the spherical harmonic deviations of the 'fluid earth' and the 'solid earth' by which the actual earth can be most approximately replaced. It is concluded that in case the rigidity were that of steel the height of the tide would be reduced to about $\frac{2}{3}$ of what it would be in case the earth were perfectly rigid, and that if the rigidity were that of glass the height of the tide would be reduced to about $\frac{2}{5}$ of what it would be in case the earth were perfectly rigid. From the observations that have been made Professor G. H. Darwin, whose method and conclusions are given in the same volume, deduces that the actual amount of the observable fortnightly tide cannot be much less than $\frac{2}{3}$, and certainly cannot be nearly so small as $\frac{2}{5}$ of that calculated in the ordinary equilibrium theory in case the earth is regarded as perfectly rigid, thus confirming Sir William Thomson's estimate of the 'tidal effective rigidity' of the earth, that it is much greater than that of glass and probably about that of steel.

It appeared to me to be worth while to try to find out what would be the effect of supposing the material of the solid replacing the earth to have finite compressibility as well as rigidity. For most solids that have been tested by experiment the two elastic constants, the resistance to compression, k, and the resistance to distortion, n, are connected very nearly by the relation $3k = 5n$, so

that in replacing the earth by an elastic solid mass of gravitating matter we may perhaps be enabled to estimate better the amount of the tidal distortion if we assume this relation to hold. I have therefore solved the following problem—A mass of solid matter, homogeneous in the natural state, free from all applied forces, and filling a spherical surface, being given, such solid is strained by its own gravitation according to the Newtonian law, and by the application of external disturbing forces having a potential expressible in spherical harmonic series. Supposing the deformed free surface expressed as a harmonic spheroid, it is required to find the amount of the harmonic inequality.—The notation of Thomson and Tait's *Natural Philosophy* is used and the solutions of the parts of this problem there considered are adopted. The problem has not been previously solved with the generality here considered.

Now the general equations of equilibrium of the solid are three, of the type

$$m\frac{\partial\delta}{\partial x}+n\nabla^2\alpha+\rho X=0 \;\ldots\ldots\ldots\ldots\ldots\ldots (a),$$

and it is well known that the solution consists of two parts, (1) any set of particular integrals of these equations, and (2) such complementary functions satisfying a system identical with (a) when the terms depending on the bodily forces such as X are left out as with the set of particular integrals (1) will satisfy the condition that the external surface remains free from stress after the deformation. The complementary functions for our problem are given in Thomson and Tait, Art. 736, and the method here used for obtaining the particular integrals is the same as that of their Art. 834, but the surface conditions cannot be immediately written down by their method. This happens for two reasons—firstly, because the stress arising from the attraction of the mass is so great compared with the other stresses, that its amount has to be estimated at the external *deformed* surface and not at the mean spherical surface as the others may be, and secondly, because part of the system of bodily forces consists in the attraction of the harmonic inequalities whose expression involves the complementary functions. It is however easy to surmount the latter difficulty by forming an equation giving the potential of the harmonic inequalities in terms of the disturbing potential and quantities that occur in the expression of the complementary functions. The method of estimating the surface tractions that must be regarded as applied to the mean sphere in consequence of the attraction of the mass, I have considered in a previous paper (*Proc. Lond. Math. Soc.*, XIX. pp. 185 sq.) in the case of vibrations, and a like method applies here. The total surface traction arising from complementary functions and particular integrals is thus found and resolved into

three components parallel to the coordinate axes, and each of these is equated to zero. The result is the determination of all the unknown harmonics occurring in the complementary functions, i.e. the complete solution of the problem, and in particular an expression is obtained for the amount of the harmonic inequality.

It is of some importance to notice the way in which gravity rigidity, and compressibility, occur in the result, and I shall for simplicity of statement here limit the expression of the results to two cases. In the first, which is that considered by Thomson and Tait, the matter is supposed incompressible. In the second the matter is supposed compressible as well as rigid in such a way that the constants m and n are connected by the relation $m = 2n$, equivalent to the relation $3k = 5n$ above referred to. In both the density is supposed equal to the earth's mean density, and the disturbing forces derivable from a potential, which is a spherical solid harmonic of order 2, say W_2. This is the case for tidal disturbing forces. Then in both cases the amount of the harmonic inequality is expressible in the form $\epsilon W_2/g$ where g is the value of gravity at the surface, and the number ϵ is a rational function of a certain number ϑ such that $(3\vartheta)^{-\frac{1}{2}}$ is the ratio of the velocity of waves of distortion in the material to the velocity due to falling through a height equal to half the radius of the sphere under gravity kept constant and equal to that at the surface. In the first case I find

$$\epsilon = \frac{15\vartheta}{6\vartheta + 19} \quad \text{..........} (b),$$

agreeing with Thomson and Tait's result, and in the second case

$$\epsilon = \frac{\vartheta}{70 + 9\vartheta} \cdot \frac{3356500 + 863100\vartheta + 55485\vartheta^2}{53900 + 27160\vartheta + 2601\vartheta^2} \quad \ldots (c).$$

The value of ϑ is zero when the matter is perfectly rigid, about $\frac{3}{2}$ when the rigidity is that of steel, about 5 when the rigidity is that of glass, and infinite when there is no rigidity at all.

We may regard ϵ as an ordinate and ϑ as an abscissa and trace the curves (b) and (c). The curve (b) is a rectangular hyperbola, and the asymptote $\epsilon =$ constant gives the limiting value $\frac{5}{2}$ of ϵ when the rigidity vanishes. The branch of the curve (c) which passes through the origin lies very near to the corresponding branch of the curve (b) for all positive values of ϑ. It has an asymptote giving the limiting value of ϵ something less than $\frac{5}{2}$. For all values of ϑ lying between $\vartheta = 0$ and $\vartheta = 5$, the curve (b) lies below the curve (c) but the distance is very minute. For some value greater than $\vartheta = 5$ the curve (b) crosses the curve (c) and the value of ϵ given by supposing the matter incompressible is greater than that given by supposing $3k = 5n$. The value of ϵ for steel $(\vartheta = \frac{3}{2})$ is about ·803 as given by the curve (b) and about ·856 as given by (c),

and for glass ($\vartheta = 5$) the value of ϵ is about 1·53 as given by (*b*) and about 1·54 as given by (*c*). To compare the heights of the tides in these cases with those given by the ordinary equilibrium theory we have to multiply by $\frac{2}{5}$, the result is the fraction by which the height of the tide is reduced by the elastic solid yielding. It is as in the case considered by Thomson and Tait about $\frac{1}{3}$ when the rigidity is that of steel, and about $\frac{3}{5}$ when the rigidity is that of glass.

May 12, 1890.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

The following communications were made:

(1) *The action of Nicotin upon the Fresh-water Crayfish.* By J. N. LANGLEY, M.A., F.R.S., Trinity College.

When 1 to 3 mgs. of nicotin, in a 1 p.c. solution, are injected beneath the epidermis of a crayfish, there is a very remarkable paralysis of certain functions of the nervous system. Almost immediately after the injection there is a tetanic contraction of the striated muscles of the body, so that the eye-stalks are drawn in, the antennæ and antennules bent backwards, the ambulatory legs are flexed, the claws firmly closed, and the tail bent. This tetanic condition lasts for a minute or two only. Before it passes off, or a little after it has done so, there is a strong peristalsis of the intestine with a rhythmic movement of the vent; the duration of this is variable, it may go on for an hour or more; for a time after it has ceased the movement of the vent can be readily produced by local stimulation. But neither the tetanic contraction of the striated muscles nor the rhythmic movement of the unstriated muscles are effects peculiar to nicotin; many other substances have a similar action.

After the tetanic period, there follows a period of complete flaccidity of all the striated muscles, the respiratory movements have ceased, and there is no movement of any segment; reflex movements are for a short time abolished, but soon slight sluggish reflex movements of flexion or extension may be obtained from any one of the ambulatory legs by scratching the shell of the leg; the movement is usually local, though sometimes the segment above and the segment below also move.

In a quarter of an hour or more,—the time varying with the amount of nicotin given and with the condition of the crayfish—the normal movements of the scaphognathites begin again, about the same time the flagella start their active, lashing movements, and, though as a rule somewhat later, there are fairly normal rhythmic movements of the swimmerets. Each of the three structures

mentioned has at irregular intervals periods of rest, these are most frequent and longest in the case of the swimmerets; least frequent and shortest in the case of the scaphognathites. The movements can be stopped for a few seconds by various slight sensory stimuli.

Although the scaphognathites, flagella, and swimmerets thus rapidly recover their ordinary rhythmic action, the crayfish lies as it is placed and there is an almost complete absence of other movement; there may be a slight slow movement of an ambulatory leg, but this is probably of a reflex nature. This condition is, in its chief features, maintained for one to two months. During this time there is a progressive slight improvement in the reflexes so that stimulation of any one segment causes a greater local effect and gives rise more readily to movements in other segments. The reflex movement however never becomes very active. The last segmental reflex to recover is the closure of the great chelæ on touching their inner surfaces; there are some other odd points in the order and extent of the recovery of the reflexes, but as these have not been quite the same in all cases, I shall consider them at a later time. If food is pushed into the œsophagus, it is apparently carried on and digested, but I have not seen any recovery in the power of the maxillæ or mandibles to aid in taking up food; in one case only (five to six weeks after nicotin injection) did the chelæ of the first two ambulatory legs close on placing a small piece of food between them, but even then no movement was made to carry the food to the mouth.

Two months is the longest time for which I have kept a crayfish after it has received 2 or 3 mgs. of nicotin; in every case there was some cause, such as the cessation of the water supply, to which death may have been due. I am, then, unable to say whether or not the above-mentioned amount of nicotin is eventually fatal. In the paralysed state of the crayfish, fungus rapidly grows on the animal, and this unless frequently removed is sufficient to cause death.

The paralysing action of nicotin upon the crayfish is due to its affecting the central nervous system, and not to its affecting either the peripheral nerves, the nerve-endings, or the muscles. This can be shown readily by giving nicotin to a crayfish and then stimulating one of the peripheral nerves, the stimulation causes contraction in the muscles of that segment.

It is clear also that the action of nicotin on the central nervous system is to an extraordinary degree selective, that is to say, that certain parts of the central nervous system are affected very much more than others; thus the nervous action causing rhythmic movements of the scaphognathites, flagella and swimmerets soon returns to its normal condition, whilst the nervous action leading to walking, swimming, masticatory movements, etc. are stopped

for a month or two if not permanently. Broadly speaking those actions which may be considered voluntary are the most interfered with by nicotin.

Further, a comparison of the effects of nicotin on the crayfish and of the effects of section of different parts of the central nervous strand, such as those made by Ward, makes it probable that in each ganglion there is a part which is paralysed by nicotin, and a part which is only slightly affected; the former being concerned with the more complicated movements, the latter with the simplest kind of reflex action.

On account of the lasting action of nicotin on certain functions of the ganglia, i.e. probably on certain nerve-cells, it is not unlikely that a morphological change may be visible in some of the nerve-cells of each ganglion, if the ganglia be observed some time after nicotin has been given. But as this point requires a careful study of the normal structure of the ganglia of the crayfish, I reserve it for a later communication.

Electrical stimulation of the nervous chain in a crayfish a short time after nicotin has been given to it differs in its effect from that produced in a normal crayfish; the contraction is less, and is limited to one or two segments above and below the point of stimulation; apparently each ganglion sends fibres to two or three segments, and the effects observed are due to a stimulation of these nerve-fibres after they have left the ganglion; whereas the other ascending and descending nerve-fibres end in nerve-cells in the various ganglia; where in consequence of the action of nicotin, the nervous impulse set up in the fibres is stopped.

(2) *On a new species of* Phymosoma. By ARTHUR E. SHIPLEY, M.A., Christ's College.

During a visit to the Bahama Islands, Mr Weldon was fortunate enough to find three specimens of a large brown *Phymosoma*, whilst investigating the Fauna of the Bimini lagoon. He came to the conclusion that these specimens belonged to no described species of *Phymosoma*, and was good enough to hand them over to me for description. I propose to call this species *Phymosoma Weldonii.*

The length of the three specimens varied between 3·5 cm. and 3 cm.; their bodies are plump and slightly curved. The ground colour of the preserved specimens is light yellow, but this is modified over the surface of the body by dark brown papillæ. In all three specimens the introvert is retracted, and in this condition is about 1 cm. long. The papillæ are of two kinds, flat, brown, rectangular, low elevations on the skin of the trunk, and conical, elevated protuberances of a light colour on the introvert.

No hooks or traces of hooks were found on the introvert.

At the base of the introvert, just behind the head, is a well-developed collar, such as I have described in detail in *Phymosoma varians.*

The mouth is surrounded by a vascular lip, which at the dorsal middle line is continuous with the base of the lophophor. The latter is in the shape of a double horseshoe, and is composed of from 70 to 80 tentacles.

There is nothing to call for remark in the arrangement of the internal organs, with the exception of the fact that there are only two retractor muscles. Such an arrangement is only met with elsewhere in *Ph. Rüppellii* from the Red Sea. The absence of hooks and of any traces of them is striking, but it occurs in five other species out of a total of 28 described.

Habitat; the Bahama Islands, Bimini lagoon.

(3) *On the action of the Papillary Muscles of the Heart**. By J. George Adami, M.A., M.B., Christ's College.

From time to time during the last twenty years continental observers have suggested that, in order to explain certain clinical phenomena, the papillary muscles of the ventricles must be looked upon as either contracting in a different manner to, or later than the rest of the heart, and in this country Ringer has suggested that one form of irregularity of the heart's action is due to a want of synchronism between the contractions of the muscles in question and the ventricular wall. On *à priori* grounds it would seem most unlikely that the papillary muscles contracted absolutely synchronously with the ventricular wall, for, were this the case, they would apparently nullify themselves. Attached as they are by the chordæ tendineæ passing from their apices to the edges and under-surface of the flaps of the auriculo-ventricular valves their main use is to aid in the complete closure of these valves, and thus to prevent regurgitation of blood into the auricles when the ventricles contract. And such is their position with relation to the auriculo-ventricular orifices that did they rapidly shorten contemporaneously with the sharp beginning of the general ventricular contraction, at a time, this is, when the blood pressure in the ventricular chamber has not been greatly raised, in the absence of such sufficient counteracting pressure upon the under surface of the valves they would the rather pull the flaps of the valves apart, and aid regurgitation. Yet up to the present no one has to my

* This paper embodies the results of a research that Professor Roy and I have been engaged upon conjointly during the last year. Fuller details will be found in a series of articles upon the 'Heart-beat and Pulse-wave,' published in the *Practitioner.*

knowledge endeavoured to obtain a simultaneous record of the work performed by these two portions of the cardiac muscle—nor has the probability that the two have different periods of action become in any way a part of general medical doctrine. Doubtless physiologists and medical men have voluntarily neglected to attempt exact observations upon this subject, from their knowledge that it is almost impossible satisfactorily to solve what would seem to be a much more simple matter,—the relation in time of the contraction of the muscle of the base and of the apex of the ventricle. And thus it has come to pass that nothing has been accomplished in what Professor Roy and I now find to be a field for research yielding suggestive, if not rich and important results.

Without the aid of diagrams it would be difficult and tedious to describe fully the apparatus employed by us in our work upon this subject, but some idea may be given of the mechanism of our instrument. In order to understand as fully as possible the nature of the papillary contraction it is necessary to gain at the same time a tracing of the contraction of the ventricular wall, so that the relation in time of the different portions of either curve obtained may be interpreted in terms of the other. We found that the most satisfactory means of recording the ventricular wall contraction was as follows. Taking two points upon the anterior surface of the left ventricle, one near to the apex, the other nearer to the base, to the former was attached a light but firm rod, moving easily upon an axis so as not to prevent or modify the general to-and-fro movement of the heart; into the latter was inserted a hook having a long shank, and these two parts of the instrument were connected in such a way with each other, and by means of a fine thread with the recording lever, that approximation of the two points upon the cardiac surface, that is to say, contraction of the muscle of the ventricular surface, resulted in an upward movement of the lever point; separation and cardiac dilatation, in a downward movement.

If now the light rod forming one limb of the above arrangement were attached to the surface of the ventricle over the region of insertion of one of the papillary muscles, a record of the contraction of this muscle could be simultaneously obtained by fixing on to the rod a cross-bar having at its further extremity a pulley, round which passed a fine thread, attached at one end to a second lever, and at the other to a strong hooked wire pulling upon one of the flaps of the mitral valve, and by this means upon the papillary muscle. To gain this attachment it was necessary to clamp off temporarily the left auricular appendix from the rest of the auricle, to make a small incision through its walls, to pass into this incision the hooked wire, to ligature the collar, in which the wire worked, to the wall of the appendix, and

then, removing the clamp, to pass the wire down through the mitral orifice and hook it over the edge of the mid-portion of one of the valve flaps. This operation required a certain amount of practice; the form of the curve shewed when it had been rightly performed. Attached in this way the hook pulled upon the free edge of the valve, and so through the chordæ tendineæ upon the papillary muscles, while the wire moved freely to and fro through the collar inserted in the auricular wall; there was little or no disturbance by clotting. The two levers recorded simultaneously upon the revolving drum, the one the contraction of the ventricular wall, the other the approximation and separation of the base of the columna carnea and the edge of the mitral valve, that is to say, the contraction and expansion of the papillary muscle.

Tracings so obtained shewed that the papillary muscles begin to contract and pull upon the mitral valves at a very definite interval after the commencement of the ventricular systole, indeed the interval between the two is so well marked that the papillary curves frequently exhibit an initial depression, due it would seem to an actual stretching of the papillary muscles and separation of their bases from the edges of the valve, the increased blood pressure within the ventricular chamber acting upon the valve and tending to drive it upwards into the auricular cavity. This is followed by the rapid contraction of the papillary muscles, which in its turn affects the curve obtained from the ventricular wall. Up to the moment when the papillary contraction begins, the lever point registering the heart-wall curve had ascended rapidly and in an almost straight line; but now the ascent is slowed, and at times there may be a slight depression or actual notch on the upstroke. This does not indicate that there has been an interference with the act of *contraction* on the part of the heart-wall, but that the *shortening* of the contracting muscles has been interrupted, and this lessened shortening is due to the sudden increase in the intra-ventricular blood pressure consequent upon the papillary contraction. Small in bulk as they are compared with the heart-wall, the musculi papillares by pulling upon the large flaps of the mitral valve must exert an influence upon a very considerable proportion of the surface of the ventricular cavity, and their contraction must have a distinct effect upon the intra-ventricular blood pressure. That this is the case is rendered evident by a comparison of the heart-wall curve with the curve of intra-ventricular pressure. The slowing or depression upon the upstroke of the former corresponds in time with a very well-marked rise or secondary wave upon the latter, and it is this sharp rise of pressure due to the contraction of the papillary muscles that hinders for the time the shortening of the muscle fibres of the wall of the ventricle: these fibres suddenly receive, as it were, an extra load,

and though there is no stoppage in the act of contraction the rate and extent of their contraction are in consequence diminished.

The sudden powerful contraction of the musculi papillares is followed by a stage in which the shortening of the muscles is slowed and the ascent of the curve more gradual: and simultaneously there is a more rapid ascent of the heart-wall curve. After this both portions of the ventricle remain for a comparatively long period in a state of contraction unaccompanied by further shortening, and the summits of both tracings are more or less flattened. We then find that the papillary muscles begin to expand before the rest of the ventricle.

To sum up the above details: *the papillary muscles begin to contract later than the ventricular walls, and commence their expansion at an earlier period. They act indeed only during that period when upon* à priori *grounds we should expect them to be contracted, not pulling upon the segments of valve until these have been brought into firm apposition by the increased blood pressure, beginning to act also at a time when further increase of pressure would tend to drive the segments upwards into the auricle, and so cause regurgitation.* Their contraction produces a sudden definite increase in the intraventricular blood pressure, well marked upon the blood pressure curves, and this increase causes a diminution in the rate of shortening of the muscle of the heart-wall, indicated by a depression upon the line of ascent of the curve obtained from the ventricular wall.

We hesitate to offer any explanation of this virtually independent action of the papillary muscles: we can only declare that the more we have studied the tracings obtained under various conditions, the more we have been led to conclude that the moment when they begin to contract is not primarily dependent upon the moment of commencing ventricular contraction. We find for example that an overdose of liquor strychnine may lead to complete asynchronism between these two components of the ventricular action; or, again, there may be a ventricular systole unaccompanied by papillary contraction, or *vice versa*. Again, the first effect of strophanthus is to cause rapidly increasing force of the contraction of the papillary muscles as compared with the heart-wall. Further, the period of papillary contraction bears no direct relation to the moment of origin of the pulse wave, to the time that is when the blood begins to pour from the heart into the arteries. Yet under normal conditions the pulse wave would seem to begin almost at the moment when the musculi papillares exert their first sharp strong pull upon the valves, and so act as an additional factor in raising the intraventricular pressure above that in the large arteries. In short, the phenomena of the papillary contraction would appear to supply further proof

as to the automatic, non-nervous action of cardiac muscle-fibres. No nerve ganglia, and, as far as we know, no nerve fibres have been made out as controlling the musculi papillares, and the moment at which they begin to contract, the duration, and the extent of their contraction would appear to be determined in large measure by the intraventricular blood pressure and the quality of the blood.

In conclusion, a few words may be said as to the way in which our observations throw light upon certain peculiarities of the pulse curve, which so far have been very variously explained—and as to which there has been much uncertainty. In tracings of the normal pulse gained by Marey's sphygmograph, or the equally unsatisfactory modification thereof usually employed in this country, or again by Dudgeon's sphygmograph in what may be termed its lucid intervals, there can often be seen two well-marked secondary waves in the first part of the curve previous to the dicrotic notch. The first of these has received the name of 'apex' or 'percussion' wave, the second that of 'tidal' or 'predicrotic.' That the former is not simply due to inertia is shewn by the fact that not unfrequently a small inertia wave may be superposed upon it, removable by proper adjustment of the instrument. By comparing the curves of the contraction of the ventricular wall or of the intraventricular blood pressure with the pulse curve taken simultaneously at the base of the aorta, Professor Roy and I have been enabled to shew that the first of these curves corresponds in time to the first period of contraction of the papillary muscles and the consequent increase in the intra-cardiac (and intra-arterial) blood pressure. This should therefore be termed the *papillary wave.* The second we consider is not by any means a secondary wave, but is really the latter portion of main wave due to the general ventricular systole, the first smaller papillary wave being superposed upon its first portion. This we would call the *systole remainder wave,* or, more shortly, *remainder wave.*

The same series of observations has also given us an explanation of the form of pulse usually termed the *anacrotic,* in which there is a small well-marked wave upon the upstroke of the pulse tracing, not, as in normal conditions, forming the apex of the curve. This form is to be found in cases where there is high intra-arterial pressure, or obstruction to the onward flow of the blood. Where there is high intra-arterial pressure there also the intra-cardiac pressure must be raised to a correspondingly high point before it becomes greater than that in the aorta, and before the valves be thrown open, that is to say, the pulse wave must begin at a later period of the cardiac systole. I have already stated that there is no absolute relation between commencement of the papillary contraction and the moment of opening of the aortic valves.

Hence in this case a fair portion of the papillary contraction has taken place before the blood begins to pass from the ventricle, or to speak more correctly, before the pulse wave can be propelled along the aorta, consequently the papillary contraction is shewn but incompletely upon the pulse wave, only its latter part is represented in the pulse, the papillary wave appears at a lower point on the ascent of the curve than under normal conditions, the greater portion of the blood being expelled by the long continuing systolic contraction; in fact, the *papillary* factor of the pulse is small, the *systolic remainder* considerable. It is interesting to note that where the intra-arterial pressure is greatly increased the intra-cardiac pressure curve shows the same tendency toward anacrotism; the papillary wave, instead of forming the apex of the curve, may be comparatively low down upon the line of ascent.

(4) Mr S. F. HARMER exhibited some living specimens of a Land-Planarian (*Rhynchodemus terrestris*, O. F. Müller) found in Cambridge. This animal was first described as a native of England by Rev. L. Jenyns (*Observations in Natural History*, London 1846), who discovered it in abundance in the woods of Bottisham Hall, near Cambridge. In the present instance, a search (made by kind permission of R. B. Jenyns, Esq.) in the same locality resulted in the discovery of a few specimens; and it was ascertained subsequently that *R. terrestris* is by no means uncommon in Cambridge (King's College, Botanic Gardens). It may readily be found by examining the damp lower surface of logs of wood which have been lying for some time on the ground. Since the first discovery of the animal in England, it seems to have been very seldom found: but from its wide distribution in Europe generally and in England, and from the fact that it is not very likely to be found unless it is specially looked for, it is probable that this animal is much commoner than is usually supposed. Several egg-capsules of *R. terrestris* were discovered on May 15, on examining fragments of rotten wood among which some specimens of the animal had been kept for a week.

May 26, 1890.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:

(1) *On Solution and Crystallization.* III. *Rhombohedral and Hexagonal Crystals.* By Prof. LIVEING.

(*Abstract.*)

In a former communication (Trans. Vol. XIV.) the author had suggested, in order to account for hexagonal crystals, an arrangement of molecules defined by supposing space to be divided by planes into right triangular prisms, and a molecule placed at each corner of the prisms. No mechanical reason, however, was forthcoming to account for the molecules assuming such an arrangement. On the other hand if we suppose the excursions of the parts of the molecule to be comprised within an ellipsoid of definite dimensions for each kind of matter, and suppose that in passing into the crystalline state these ellipsoids pack themselves as closely as possible (which they will do if they attract each other according to any law), the surfaces of minimum tension will be certain planes having the symmetry observed in crystals. That symmetry depends on the dimensions of the ellipsoids. If they be spheroids, and if oblate have their axes in any ratio except 2 : 1, and if the orientation of the axes be such that each spheroid is touched by six others in points lying in a plane perpendicular to the axis of symmetry, the arrangement will be the same as if space were divided into equal rhombohedra and a molecule placed at every corner. In this case the surfaces of minimum tension will be planes having a rhombohedral symmetry. But in arranging the spheroids so as to place the greatest number in a given space there are two arrangements with the centres in planes perpendicular to the axis of symmetry which give the same number of spheroids per unit of volume, and are therefore so far equally probable. These two arrangements correspond to twin crystals when the twin axis is the axis of symmetry. A crystal formed of such molecules may therefore, so far as the packing of the molecules determines its structure, be built up of alternate layers, of no particular thickness, of twin crystals. If the external form be rhombohedral such alternations will in general give ridged faces, for which the surface tension will not be the minimum. But if the external form be any one (or more) of those known as hexagonal, it will be identical for the two individuals of the twin, and this circumstance will determine the growth of hexagonal forms in cases in which the surface tension of the hexagonal forms is not much greater than that of rhombohedral forms. At the same time

the fact that hexagonal forms lend themselves to the production of approximately globular masses with a minimum total surface, will increase the tendency to the development of hexagonal forms. We may still however get rhombohedral forms in some cases. It has been shown that if the molecular ellipsoids be oblate spheroids with their principal diameters in ratio of $\sqrt{2} : 1$, and they be arranged so that each be touched by four others in the plane of its principal circular section and by four others in each of two other planes parallel to the first, the crystal will have cubical symmetry, and if we suppose the system uniformly strained in the direction of one of the diagonals of the cubes the spheroids will be deformed into ellipsoids and the crystal will have rhombohedral symmetry. In this case the alternations of twins will not be equally probable and the rhombohedral forms will generally predominate. Formulæ are given for calculating the relative probability of different forms when the angular element of the crystal is known, and their application shown by examples.

(2) *On the Curvature of Prismatic Images, and on Amici's Prism Telescope.* By J. LARMOR, M.A., St John's College.

It is well known that, in homogeneous light, a prism acts as a telescope in magnifying transversely the dimensions of objects in the field of view, while their longitudinal dimensions remain unchanged. In fact an incident parallel beam emerges as a parallel beam, so that the prism forms a telescopic system; and the transverse magnification along any ray is, by the general law applicable to such systems, equal to the inverse ratio of the breadths of these incident and emergent pencils.

These considerations apply equally to any battery of prisms.

As a single prism has a position of minimum dispersion which is different from the position of minimum deviation, it follows that two prisms, of the same kind of glass, may be combined in opposing fashion so as to form an achromatic pair, while some deviation remains, and therefore also some magnification. By combining in perpendicular planes two such achromatic doublets, of equal magnifying power, Amici long ago succeeded in producing a telescope which magnified equally (about 4 times) in all directions, was made of the same kind of glass throughout and yet achromatic, and, according to Sir John Herschel's experience of it (*Encyc. Metrop.* 'Light,' § 453), gave images of remarkable perfection.

The image of a straight-edge or slit, seen through a prism without a collimator, is a curved arch or bow: and it has been pointed out by Sir Howard Grubb that when the prism is rotated the curvature of this arch is proportional to the dispersion of the

spectrum produced. This law will be formally established below for any cylindrical optical system whatever which is composed throughout of the same kind of glass: it may be readily verified by a cursory examination of a pair of prisms standing on a flat plate. It follows from it that each of Amici's doublets gives images of straight lines which are free from curvature for the very reason that they are free from chromatic defect; and the remarkable absence of distortion noticed by Sir John Herschel is explained.

We proceed to obtain a formula for the curvature of the image of a vertical slit seen a distance a through a system of prisms and cylindrical lenses of the same material, standing on a horizontal plane. The horizontal projection of a ray which is travelling at an inclination θ to the horizontal plane—and is therefore refracted to an inclination θ', given by the same law $\sin\theta = \mu \sin\theta'$ as that of Snell—will be refracted according to the variable index $\mu \cos\theta' / \cos\theta$, or approximately $\mu + \frac{1}{2}(\mu - \mu^{-1})\,\theta^2$, as θ is small. This principle, as was pointed out by Stokes, will suffice for the solution of the problem.

Thus the coordinates of a point on the image are

$$x = a\theta,$$

$$y = a\frac{dD}{d\mu} \cdot \tfrac{1}{2}(\mu - \mu^{-1})\,\theta^2,$$

where D is the deviation of a horizontal ray.

The curvature is equal to $2y/x^2$, and is therefore

$$\frac{\mu - \mu^{-1}}{a}\frac{dD}{d\mu},$$

wherein $\frac{dD}{d\mu}$ is clearly the angular dispersion of the spectrum produced by the combination.

The investigation is no more difficult for a slit inclined at an angle ϵ to the vertical. In this case

$$x = a\theta,$$

$$y = a\theta\tan\epsilon + a\left\{\frac{dD}{d\phi}\,\theta\tan\epsilon + \frac{dD}{d\mu}\,\tfrac{1}{2}(\mu - \mu^{-1})\,\theta^2\right\};$$

therefore

$$y - \tan\epsilon\left(1 + \frac{dD}{d\phi}\right)x = \frac{1}{2a}\frac{dD}{d\mu}(\mu - \mu^{-1})\,x^2.$$

Thus the image is parabolic, of the same curvature

$$\frac{\mu - \mu^{-1}}{a}\frac{dD}{d\mu},$$

as above; and

$$\frac{\tan \eta}{\tan \epsilon} = 1 + \frac{dD}{d\phi},$$

where η is the inclination of the image to the vertical, and ϕ is the angle of incidence of the axial ray on the first face.

(3) *On some theorems connected with Bicircular Quartics.* By R. LACHLAN, M.A., Trinity College.

The object of this paper is to extend to Bicircular Quartics, and twisted quartics, Sylvester's theory of residuation in connection with the plane cubic.

1. A curve of the $2n$th order, having multiple points of the nth order at each of the circular points, may be called a circular curve of the $2n$th order. Such a curve is determined by $n(n+2)$ points, as is seen at once by writing down its equation; and two circular curves of the $2n$th and $2m$th orders intersect in $2mn$ points, other than the circular points.

Let U, V be any two circular curves of the $2n$th order passing through $n(n+2)-1$ given points, then $U+kV$ will represent any circular curve of the $2n$th order passing through these points, but such a curve must obviously pass through all the points in which U and V intersect. Hence it follows that any circular curve of the $2n$th order which passes through $n(n+2)-1$ fixed points must pass through $(n-1)^2$ other fixed points.

2. Further we may show that every circular curve of the $2n$th order which passes through $2np-(p-1)^2$ points on a circular curve of the $2p$th order (p being $< n$) meets this curve in $(p-1)^2$ other fixed points. For if we draw any circular curve of the $2(n-p)$th order through $(n-p)(n-p+2)$ assumed points, then since

$$2np-(p-1)^2+(n-p)(n-p+2) = n(n+2)-1,$$

any circular curve of the $2n$th order which passes through the given $2np-(p-1)^2$ points on the curve of the $2p$th order and also through the assumed points on curve of the $2(n-p)$th order, must pass through $(n-1)^2$ other fixed point. But the given curve of the $2p$th order and the curve of the $2(n-p)$th order make up one such circular system; hence these $(n-1)^2$ points must lie on one or other of these curves of lower order. The most that can lie on the curve of the $2(n-p)$th order is

$$2n(n-p)-(n-p)(n-p+2), \text{ i.e. } (n-1)^2-(p-1)^2;$$

hence the remaining $(p-1)^2$ points must lie on the curve of the $2p$th order. Hence the truth of the theorem enunciated is manifest.

3. An important particular case may be thus stated: Every circular curve of the $2n$th order which passes through $4n-1$ fixed points on a bicircular quartic must pass through one other fixed point*.

4. From this theorem we may deduce the following: If of the $4(m+n)$ intersections of a circular curve of the $2(m+n)$th order with a bicircular quartic, $4m$ lie on a circular curve of the $2m$th order, the remaining $4n$ lie on a curve of the $2n$th order.

For let U_m denote the curve of the $2m$th order, and let a curve U_n of the $2n$th order be described passing through $4n-1$ of the remaining $4n$ points; then these curves U_m, U_n together make up a circular system of the $2(m+n)$th order passing through $4(m+n)-1$ points on a bicircular quartic, this curve must therefore pass through one other fixed point, which must clearly lie on U_n; also this point must be a point in which the given curve of the $2(m+n)$th order meets the bicircular quartic; hence we see that the theorem stated above must be true.

5. If now we have two systems of points α, β, which together make up the complete intersection of a bicircular quartic with a circular curve of any degree, i.e. if $\alpha+\beta$ is a multiple of 4, one of these systems may be called the *residual* of the other. Since through a given system of points, any number of curves of different orders may be described, it is evident that a given system of points α has an infinite number of residual systems β, β', &c. Two systems of points β, β', may be called *coresidual* systems if both are residuals of the same system α.

6. We have at once the following theorems:

i. Two points which are coresidual must coincide. This is merely a restatement of the theorem in § 3; for if through $4n-1$ points α on a bicircular quartic we describe two circular curves of the $2n$th order, meeting the quartic again in the points β, β', then β and β' are one and the same point.

ii. If two systems β, β' be coresidual, any system α' which is a residual of one will be a residual of the other.

Suppose that through any system α, two curves U_p, U_q are described meeting the quartic again in systems β, β', then by definition β, β' are coresidual systems; then if through β' a curve

* Some interesting results connected with bicircular quartics, obtained by developing this theorem, were given in a paper communicated to the London Mathematical Society in May, 1890.

U_r be drawn meeting the quartic again in the system of points α', then the systems β, α' will also be residual. For since the systems α, β make up the intersection of a curve U_p with the quartic, and α', β' make up its intersection with a curve U_r, the four systems together make up the intersection with the quartic of a curve whose order is $2(p+r)$; but the systems α, β' together make up the intersection of the quartic with the curve U_q of order $2q$, and therefore by § 4, the systems α', β together make up the complete intersection of the quartic with a curve whose order is $2(p+r-q)$.

iii. Two systems which are coresidual to the same are coresidual to each other.

If β and β' are coresidual as having a common residual α, and if β', β'' have a common residual α'; then by the last theorem α is also a residual of β'', and α' a residual of β; that is, if β, β'' are each of them coresidual with β', then β, β'' are coresidual with each other, for α, α' are each of them a residual of β, β''.

7. Suppose now that we have given a system of $4p+1$ points on a bicircular quartic, through them we may draw a circular curve of order $2(p+r)$ and we obtain a residual system of $4r-1$ points; through these we may draw a circular curve of order $2(r+s)$ and the residual system will consist of $4s+1$ points; through these we may draw a circular curve of order $2(s+t)$ and we obtain a residual consisting of $4t-1$ points. If at any stage where we have a residual of $4n-1$ points we draw a circular curve through them of order $2n$ we obtain a residual of a single point, and it follows from the theorems stated in § 6 that this point must be the same whatever be the process of residuation. Moreover whatever system of points we start with, either a system of $4p+1$ points or a system of $4p-1$ points, we can always by an even or odd number of stages, obtain a single point which will be a coresidual or a residual of the given system, according as the number of points in the given system is $4p+1$ or $4p-1$.

The principles just established enable us to find, by means of circular constructions, the point residual or coresidual to any given system of points, the number of which is $4p \pm 1$.

8. To find the coresidual point of a system of five given points.

Let P_1, P_2, P_3, P_4, P_5 be the points, through any three of these P_1, P_2, P_3 say, draw a circle cutting the quartic in Q_1, and through the other two P_4, P_5 draw a circle cutting the quartic in Q_2, Q_3, then the circle $Q_1Q_2Q_3$ will cut the quartic in the point R which will be the coresidual of the given system.

Let the points P_1, P_2, P_3, P_4, P_5 coincide, then we see that to obtain the coresidual of five consecutive points P, we have to draw the circle of curvature at P meeting the quartic in Q, and one of the bitangent circles at P touching the quartic again at Q_1,

then the circle which touches the quartic at Q_1 and passes through Q will cut the curve again in R the coresidual of five consecutive points at P. Hence we have the theorem that if the bitangent circles at P touch the bicircular quartic again at the points Q_1, Q_2, Q_3, Q_4 and the osculating circle at P cut the curve again in the point Q, then the four circles which can be drawn passing through Q and touching the quartic at Q_1, Q_2, Q_3, Q_4 cut the quartic again in the same point R.

Again if we draw a bicircular quartic passing through the five points P_1, P_2, P_3, P_4, P_5 it must cut the given quartic in three points p_1, p_2, p_3, the circle through which must pass through R the coresidual of the five given points; hence if we wish to draw a bicircular quartic passing through five given points on a given bicircular quartic and osculating the latter elsewhere, we have merely to draw a circle osculating the given curve and passing through R the coresidual of the five given points; but nine such circles can be drawn; hence nine systems of bicircular quartics can be drawn passing through five given points on a given bicircular quartic, which have three-point contact with it elsewhere.

9. To find the residual point of a system of seven given points.

Let the points be $P_1, P_2, \ldots P_7$; through any three of them such as P_1, P_2, P_3 draw a circle cutting the quartic in Q_1, through P_4, P_5 draw a circle cutting the quartic in Q_2, Q_3, and through P_6, P_7 a circle cutting the quartic in Q_4, Q_5. And then we may find R the coresidual of the system Q_1, Q_2, Q_3, Q_4, Q_5 as in § 8, R will be the residual of the given system of seven points.

Or we might replace any six of the points by their coresidual points; thus let the circle $P_1P_2P_3$ cut the quartic in Q_1, and the circle $P_4P_5P_6$ cut the quartic in Q_2; and then let any circle be drawn through Q_1, Q_2 to cut the quartic in P_1', P_2'; which two points constitute a coresidual system of the system $P_1, P_2 \ldots P_6$. Then if the circle P_1', P_2', P_7 cut the quartic in R, R will be the residual of the given system.

By this method we are enabled to find the eighth point in which any bicircular quartic which passes through the seven given points cuts the given quartic, for every bicircular quartic through the seven given points must pass through R. This method assumes that one quartic through the seven points is given; and thus the problem is not the same as finding the eighth point when *only* seven points are given.

10. To find the residual point of seven consecutive points.

Let the points $P_1 \ldots P_7$ in § 9 coincide with the point P, let the circle of curvature at P meet the curve in Q, also let Q_1, Q_2, Q_3, Q_4 be the points of contact of the bitangent circles at Q; then two consecutive points at Q_1 may be considered as a coresidual system

of the six points $P_1, P_2, \dots P_6$; and then the circle touching the curve at Q_1 and passing through P_7, i.e. P, must meet the curve again in R the residual point of the seven consecutive points P.

Otherwise, we may obtain R' the coresidual point of five consecutive points P as in § 8, and then the circle which touches the curve at P and passes through R' must meet the curve again in R.

Incidentally we may notice that we have the theorem: if the circle of curvature at any point P of a bicircular quartic meet the curve again in Q, and if Q_1, Q_2, Q_3, Q_4 be the four points of contact of the bitangent circles at Q, then the four circles which can be drawn touching the curve at these points respectively and passing through P, cut the curve again in the same point R.

Hence we see that R can only coincide with P, when Q_1, Q_2, Q_3, Q_4 are the points of contact of the bitangent circles at P, in which case Q must coincide with P, i.e. P must be a cyclic point.

Thus at any point P on a bicircular quartic we can in general draw other bicircular quartics having seven point contact at P, and they will all cut the curve again in the point R. If however P be a cyclic point these bicircular quartics will have eight-point contact with the given quartic at P.

11. All these theorems admit of translation so as to apply to twisted quartics, we have merely to substitute in the enunciation of any theorem the word plane for circle. In fact we have only to prove that if any surface of the nth order passes through $4n-1$ fixed points on the curve of intersection of two quadrics, it must also pass through one other fixed point.

Let U_2, V_2 denote the two given quadrics, and U_n any surface of the nth degree passing through $4n-1$ fixed points on the curve of intersection of U_2, V_2; then we have to show that any other surface of the nth degree passing through these fixed points will cut the curve of intersection of U_2, V_2 in the same point as U_n. Let a surface U_{n-2} of the $(n-2)$th order be drawn passing through $n(n-2)$ arbitrary points on the curve of intersection of V_2 with U_n, and also through $\frac{1}{6}(n-1)(n-2)(n-3)$ arbitrary points on the surface U_n; also let a surface V_{n-2} of the $(n-2)$th order be drawn passing through $n(n-2)$ arbitrary points on the curve of intersection of U_2 with U_n, and also through the same points on U_n as U_{n-2}. Then we have three surfaces U_n, $U_{n-2} \,.\, U_2$, $V_{n-2} \,.\, V_2$ each of the nth order and each passing through the same points, whose number is

$$4n-1+2n(n-2)+\tfrac{1}{6}(n-1)(n-2)(n-3) = \tfrac{1}{6}n(n^2+6n+11)-2,$$

which is two less than the number necessary to determine a surface of the nth degree, any surface of the nth degree therefore which passes through these points must be of the form

$$U_n + \lambda U_{n-2} \,.\, U_2 + \mu V_{n-2} \,.\, V_2,$$

and consequently must pass through all the points in which the three surfaces U_n, $U_{n-2}U_2$, $V_{n-2}V_2$ intersect. Hence any surface of the nth degree which passes through the $4n-1$ points of intersection of the surfaces U_2, V_2, U_n must pass through the remaining point of intersection.

12. Exactly as in § 4 we may deduce the theorem: if of the $4(m+n)$ intersections of a surface of the $(m+n)$th order with the curve of intersection of two quadrics, $4m$ lie on a surface of the mth degree, then the remaining $4n$ must lie on a surface of the nth degree.

Also the definitions of residual and coresidual systems of points require such slight modification that it is needless to recite them. And in applying the principles of residuation we see that we have merely to substitute planes for circles, and thus we obtain what might be called 'planar' constructions for finding the residual or coresidual point of any system of points on a twisted quartic.

June 2, 1890.

At a meeting of the Council of the Society, it was decided, in accordance with the Reports of the adjudicators, Sir W. Thomson, Lord Rayleigh, and Prof. G. H. Darwin, to award the Hopkins Prize for the period 1883—5 to W. M. Hicks, M.A., F.R.S., for his memoir upon the *Theory of Vortex Rings* (Phil. Trans. 1885) and for his earlier memoirs upon related subjects—also to award the Hopkins Prize for the period 1886—8 to Horace Lamb, M.A., F.R.S., for his paper on *Ellipsoidal Current-Sheets* (Phil. Trans. 1887) and for his numerous other papers on Mathematical Physics.

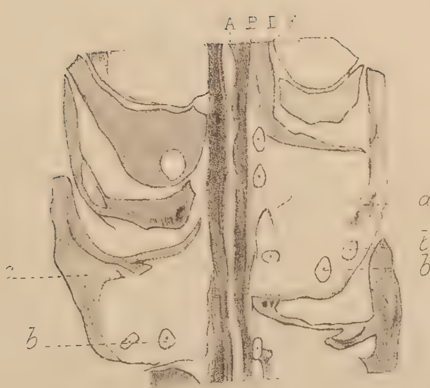


Fig. 1.

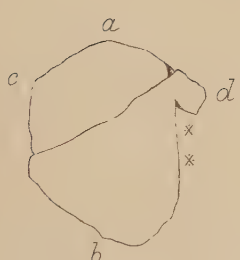


Fig. 3.

Fig. 4

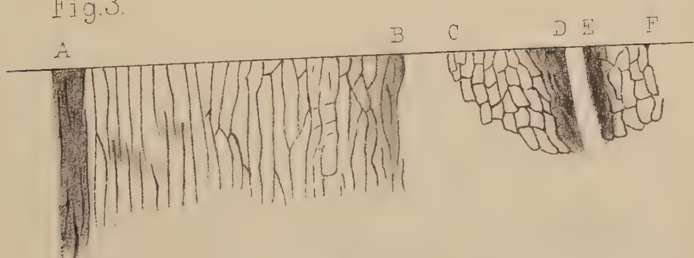


Fig. 2.

PROCEEDINGS

OF THE

Cambridge Philosophical Society.

October 27, 1890.

ANNUAL GENERAL MEETING.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

The following Fellows were elected Officers and new Members of Council for the ensuing year:

President:
Prof. G. H. Darwin.

Vice-Presidents:
Mr J. W. Clark, Prof. Babington, Prof. Liveing.

Treasurer:
Mr R. T. Glazebrook.

Secretaries:
Mr J. Larmor, Mr S. F. Harmer, Mr E. W. Hobson.

New Members of Council:
Dr A. Hill, Dr A. S. Lea, Mr A. Harker, Mr L. R. Wilberforce.

The names of the Benefactors of the Society were recited by the Secretary.

Fourteen names were proposed, with the approval of the Council, for election as Honorary Members of the Society.

The retiring President, Mr J. W. CLARK, before vacating the chair, gave a short sketch of the origin and early years of the

Society, which will be published as a separate part of the Proceedings.

The President elect, Prof. G. H. DARWIN, then took the Chair, and the following Communication was made to the Society:

(1) *On some Compound Vibrating Systems.* By C. CHREE, M.A., King's College.

(*Abstract.*)

The vibrating systems treated in this memoir are bounded either by concentric spherical or by coaxial cylindrical surfaces, and the vibrations are of those types in which the displacements are either wholly radial or wholly transverse.

By a *simple* system is meant a spherical or a cylindrical shell of a single isotropic medium; by a *compound* system is meant a stratified medium in which the surfaces separating adjacent media, or *layers*, are spherical or cylindrical according as the outer surfaces are spherical or cylindrical.

Those functions which when equated to zero constitute the frequency equations for a simple system are termed *frequency functions.* A method is developed whereby the frequency equation for a compound system of any number of layers, composed of different isotropic media, can be at once written down in a form which involves the frequency functions of the several layers.

The general result so obtained is employed in determining the change in the pitch of the several notes in an otherwise isotropic simple shell owing to the existence of a thin intercalated layer of a different isotropic medium. The dependence of the magnitude of the change of pitch on the nature of the difference between this *altered layer* and the remainder of the shell, on the position of the altered layer, and on the value of Poisson's ratio for the unaltered medium is considered for the system of notes which the vibrating system in question is capable of producing.

The law of variation of the magnitude of the change of pitch in solid spheres or cylinders with the position of an altered layer, which differs from the remainder of the system in a given assigned way, is represented by a curve or curves. Every such curve shows in a very simple manner the comparative magnitude of the largest possible changes of pitch, for all possible notes of the system, which can arise from a given alteration of material, throughout a layer of given volume or of given thickness as the case may be. The corresponding positions of the layer may also be immediately derived from the curves for all those notes of the system whose frequencies are recorded.

Tables are constructed showing the positions where a thin layer differing in an assigned way from the remainder is most effective

in altering the pitch of several of the notes of lowest pitch, and other tables show the numerical magnitude of the corresponding maximum changes of pitch.

The changes in the types of vibration in solid spheres or cylinders due to the existence of thin intercalated layers of other media are determined by a different method. This method also leads to expressions for the changes of frequency in these systems which are identical with those obtained by the first method.

The results obtained in the memoir are very numerous, and many of them seem of interest from a physical point of view. They show that general laws as to the effects of altering the *stiffness* or *elasticity* of vibrating systems unless carefully restricted may lead to very erroneous conclusions.

November 10, 1890.

PROFESSOR G. H. DARWIN, PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:

(1) *Note on the principle upon which Fahrenheit constructed his Thermometrical Scale.* By ARTHUR GAMGEE, M.D., F.R.S., Emeritus Professor of Physiology in the Owens College (Victoria University).

[*Abstract; reprinted from the Cambridge University Reporter*, *Nov.* 18, 1890.]

The author commenced by drawing attention to the fact that, although the Fahrenheit thermometer has been so generally used in England, no accurate information was to be found in our text-books concerning the principles upon which its scale had originally been constructed. He referred, however, to a view advanced by Professor P. G. Tait in his elementary treatise on 'Heat,' and which had been accepted by many teachers, according to which Fahrenheit divided his scale between 32° and 212° into 180 degrees, in imitation of the division of a semi-circle into 180 degrees of arc. This theory rested on the incorrect supposition that, before Fahrenheit's time, Newton had suggested, as a basis for a thermometric scale, the fixing of the freezing and boiling points of water, the space between these being divided into a number of equal degrees. The author pointed out that in his "Scala graduum caloris," Newton made no such suggestion as that attributed to him by Professor Tait, and prior to him by Professor Clerk Maxwell; and, indeed, that Fahrenheit had settled the basis of his scale and had constructed a large number of thermometers which were used by scientific men throughout Europe, many years before the discovery by Amanton (which Fahrenheit confirmed and gave precision to)

of the fact that under a constant pressure the boiling point of water is constant.

The author stated that the thermometers which were first constructed by Fahrenheit were sealed alcoholic thermometers, provided with a scale in which two points had been fixed. The zero of the scale, representing the lowest attainable temperature, was found by plunging the bulb of the thermometer in a mixture of ice and salt, whilst the higher of the two points was fixed by placing the thermometers under the arm-pit or inside the mouth of a healthy man. The interval between these two points was, in the first instance, divided into 24 divisions, each of which corresponded to supposed well characterized differences in temperature, and each being subdivided into four. In his later alcoholic and mercurial thermometers, the 24 principal divisions were suppressed in favour of a scale in which 96 degrees intervened between zero and the temperature of man; in these later thermometers the 32nd degree was fixed by plunging the bulb of the thermometer in melting ice.

The author then pointed out that Fahrenheit was led to construct mercurial thermometers in order to be able to ascertain the boiling point of water; with this object the scale constructed, as has been stated, was continued upwards, in some cases so as to include 600 degrees.

It was *as the result by experiment alone*, that the number 212 was obtained as the temperature at which water boils, at the mean atmospheric pressure.

The author in conclusion argued that Fahrenheit took as the basis of his thermometric scale the duodecimal scale which he was constantly in the habit of employing.

(2) *On Variations in the Floral Symmetry of certain Flowers having Irregular Corollas.* By William Bateson, M.A., Fellow of St John's College, and Anna Bateson.

(3) *On the nature of the relation between the size of certain animals and the size and number of their sense-organs.* By H. H. Brindley, B.A., St John's College.

[*Abstract; received November* 29, 1890.]

In speculation as to the evolution of various forms it is generally held as a principle, that the conditions of the struggle for existence are such that variations in the direction of atrophy or diminution in bulk of a useless organ must necessarily be beneficial by reason of the saving of tissue and effort which is effected

by this reduction. It has been assumed by many that this benefit must be so marked as to lead to the Natural Selection of the individuals thus varying. This principle has been invoked especially in the case of sense-organs, and, for example, it has been suggested that the blindness of cave-fauna may have come about by its operation.

With the object of testing the truth of this assumption, it seemed desirable to obtain a knowledge of the normal variations in size and number of sense-organs occurring within the limits of a single species. The cases chosen were (1) The olfactory organ of Fishes (Eel, Loach, *Pleuronectidæ*, &c.), and (2) The eyes of *Pecten opercularis*. In the first case tables were given shewing that large individual fluctuations occur, but that on the whole the number of olfactory plates increases with the size of the body. It was pointed out that the size of the eye in Fishes also increases with the size of the body.

In the case of *Pecten*, however, though the *size* of the eyes increases with the diameter of the animal, yet in specimens having a diameter of 3 cm.—6 cm. the *number* of the eyes is not thus related (cp. PATTEN), but varies in a most surprising and, as it were, uncontrolled manner.

Statistics were given shewing that in individuals of the same size, the number of eyes may vary between 70 and 100, and that no uniformity is to be found. It was pointed out that these eyes are large and complicated organs, having lens, retina, tapetum, &c., involving great cost in their production. These facts suggest that the "economy of growth" cannot be a principle of such precise and rigid character as to warrant its employment as a basis for speculation as to the mode of evolution of a species. The diverse results in the case of the two sets of organs examined further indicate that the problem is one of far greater complexity and shews clearly that argument from analogy is inadmissible in these cases.

(4) *On the Oviposition of* Agelena labyrinthica. By C. WARBURTON, B.A., Christ's College.

[*Abstract; reprinted from the Cambridge University Reporter, Nov.* 18, 1890.]

The oviposition and cocooning of *Agelena labyrinthica* is a striking case of the performance of a series of complicated operations in obedience to a blind instinct.

The eggs are always laid at night, but the presence of artificial light is quite disregarded by the animal.

For about 24 hours before laying, the spider is engaged in preparing a chamber for the purpose.

Near its roof a small sheet is then formed, and the eggs are laid upwards against it and are covered with silk. A box is then constructed with this sheet as its roof, and is firmly attached by its angles to the roof and floor of the chamber. This box is constructed and jealously guarded even if the eggs are removed immediately on oviposition.

The whole operation involves about thirty-six hours of almost incessant industry.

(5) *Supplementary list of spiders taken in the neighbourhood of Cambridge.* By C. WARBURTON, B.A., Christ's College.

[*Received November* 14, 1890.]

In Vol. VI. of these *Proceedings* a list was given of some hundred species of local Araneæ. To these must now be added the following, some of which have been taken since the former publication, while others are inserted on the authority of the Rev. O. Pickard-Cambridge, who has kindly furnished a list of Spiders sent to him some years ago by the late Mr Farren.

Unfortunately Mr Farren did not record the exact locality nor the frequency of his captures, but he is known to have carefully searched Wicken Fen, which is probably the habitat of most of his species.

DYSDERIDAE.

DYSDERA
crocota, C. L. Koch, rare, Castle Hill.

SEGESTRIA
senoculata, Linn.

DRASSIDAE.

DRASSUS
troglodytes, C. L. Koch, rare, Wicken Fen.
blackwallii, Thor.

CLUBIONA
corticalis, Walck., rare, University bathing enclosure.
reclusa, Cambr.

ANYPHAENA
accentuata, Walck.

PHRUROLITHUS
festivus, C. L. Koch, Fleam Dyke.

DICTYNIDAE.

DICTYNA
latens, Fabr.

AGELENIDAE.

HAHNIA
nava, Bl., rare, Wicken Fen.

LETHIA
humilis, Bl.

THERIDIIDAE.

THERIDION
simile, C. L. Koch.
tinctum, Walck., common, on shrubs and bushes.
rufolineatum, Luc.

NERIENE
cornuta, Bl., rare, in the "Backs."
nigra, Bl., rare, Turf Fen, Chatteris.
fuscipalpis, C. L. Koch. In the bathing enclosure.
apicata, Bl.
bicolor, Bl. Castle Hill.
bituberculata, Wid.

WALCKENAERA
bifrons, Bl.
unicornis, Bl.
cristata, Bl., rare, Christ's Coll. Garden.

PACHYGNATHA
listeri, Sund.

EURYOPIS
blackwallii, Cambr.

LINYPHIA
nigrina, Westr.
setosa, Cambr., Wicken Fen.
clathrata, Sund., Wicken Fen.
circumspecta, Bl., rare.

EPEIRIDAE.

EPEIRA
acalypha, Walck.
solers, Walck.
quadrata, Clrk., occasional, Fens.

ERRATA,
In the previous list, *Amaurobius fenestralis* should have been recorded as *rare* instead of *common*, and the habitat of *Theridion varians* as being "boathouses and out-buildings" rather than "bushes".

November 24, 1890.

PROF. G. H. DARWIN, PRESIDENT, IN THE CHAIR.

The following gentlemen, duly nominated by the Council, were elected Honorary Members of the Society:

FRANCESCO BRIOSCHI; on the ground of his contributions to mathematical science by his investigations in the theory of forms, the theory of equations, and in elliptic and hyperelliptic functions.

LEOPOLD KRONECKER; on the ground of his contributions to mathematical science by his investigations in the theory of numbers and elliptic functions.

SOPHUS LIE; on the ground of his contributions to mathematical science by his investigations in geometry, in the theory of differential equations, and in the theory of groups.

HENRI POINCARÉ; on the ground of his contributions to mathematical science by his investigations in the theory of functions and in mathematical physics.

George William Hill; on the ground of his contributions to astronomical science by his investigations on the secular motion of the Moon's perigee and other researches in the lunar theory.

J. Willard Gibbs; on the ground of his contributions to physical science and specially to the sciences of thermodynamics and electromagnetism.

Heinrich Hertz; on the ground of his contributions to the science of electromagnetism and specially for his brilliant experimental verification of Maxwell's theory.

Arthur Schuster; on the ground of his contributions to physical science and specially for his researches on spectrum analysis and on the passage of the electric spark through high vacua.

Victor Meyer; on the ground of his contributions to chemical science, namely his researches on the nitro compounds of the fatty series, on the thiophenes, on pyro-chemistry, his development of Raoult's researches, and many other investigations.

James Dwight Dana; on the ground of his contributions to mineralogical and geological science, namely his researches on coral islands, his great work A System of Mineralogy, and numerous other papers.

Henry Bowman Brady; on the ground of his zoological researches and in recognition of his generosity in presenting to the University a valuable collection of Foraminifera.

Rudolf Heidenhain; on the ground of his contributions to physiology, dealing with the physiology of secretion and absorption, and the physiology of muscles.

Elias Metschnikoff; on the ground of his researches in many fields of biological science, and especially in the study of embryology.

Melchior Treub, Director of the Botanical Gardens, Java; on the ground of his general researches in botany.

The following were elected Fellows of the Society:

S. Ruhemann, M.A., Gonville and Caius College.
A. W. Flux, B.A., Fellow of St John's College.
H. H. Brindley, B.A., St John's College.

The following were elected Associates:

David Sharp, M.B. (Edin.), F.R.S., Curator in Zoology.
H. Gotobed.

The following Associates were re-elected for a further period of three years:

R. Bowes,	R. I. Lynch,
J. Carter,	W. E. Pain,
A. Deck,	W. W. Smith.

The following Communications were made to the Society:

(1) *On the beats in the vibrations of a revolving cylinder or bell.* By G. H. BRYAN, M.A., St Peter's College.

IN this paper I propose to investigate the nature of the beats which may be heard when a vibrating shell in the form of a cylinder or other surface of revolution has imparted to it a rotatory motion about its axis of figure.

It might at first appear that, unless such a body were revolving with angular velocity comparable with the frequencies of the vibrations, the latter would not be affected in any sensible manner, and that the only important effect of rotation would be in permanently straining the body, owing to centrifugal force. This is, for example, the point of view taken by Mr Love in his paper on "The free and forced vibrations of an elastic spherical shell containing a given mass of liquid,"* in which the author uses the expressions for the accelerations referred to moving axes when dealing with the oscillations of the liquid, but not when dealing with those of the elastic envelope, although both are supposed to rotate together. But, while we may be justified in neglecting the effect on any single period of such small changes in the system as those due to rotation, yet the slight opposed changes produced in the periods of two similar vibration forms which travel in different directions round the shell may produce phenomena of beats, which in the case of very rapid vibrations like those of sound, are among the most noticeable effects of the rotation.

If a straight wire of circular section, clamped symmetrically at one end, be made to rotate slowly about its axis while executing transverse vibrations, it is well known that the plane of vibration will remain fixed in space instead of turning with the wire. If the vibrations are audible we shall, therefore, hear a continuous sound. In the case of a tuning-fork the plane of vibration must necessarily turn with the fork, so that beats are heard if it be rotated.

When however the vibrating body is such as a bell, rotation about its axis will produce an intermediate effect by causing the nodal meridians† to revolve with angular velocity less than that of the body, and depending in each case on the mode of vibration considered. This phenomenon, which forms the subject of the present paper, appears to be new, yet nothing is easier than to verify it experimentally. If we select a wine-glass which when struck gives, under ordinary circumstances, a pure and continuous tone, we shall on twisting it round hear beats, thus showing that the nodal meridians do not remain fixed in space. And if the observer will turn himself rapidly round, holding the vibrating

* *Proc. Lond. Math. Soc.* XIX.

† That is meridians along which the vibration has no radial component.

glass all the time, beats will again be heard, showing that the nodal meridians do not rotate with the same angular velocity as the glass and observer. If the glass be attached to a revolving turn-table it is easy to count the number of beats during a certain number of revolutions of the table, and it will thus be found that the gravest tone gives about 2·4 beats per revolution. As this type of vibration has 4 nodes we should hear 4 beats per revolution if these nodes were to rotate with the glass, we conclude therefore that the nodal angular velocity is in this case about $\frac{3}{5}$ of that of the body.

It may not, perhaps, be out of place to explain from first principles *why* the nodal meridians revolve less rapidly than the body. Take the case of a ring or cylinder revolving in the direction indicated by the arrows in figure 1, and consider the mode of vibration with four nodes, B, D, F, H. Suppose also that at the instant considered the ring is changing from the elliptic to the circular form indicated in the figure.

Owing to the rotatory motion, the points A, E where the ring is initially most bent will be carried forward and parts initially less bent will be brought to A and E. Similar remarks apply to the points C, G, where the ring is initially least bent. Hence the points of maximum and minimum curvature, and therefore, also, the nodes must be carried round in the same direction as the ring, and cannot remain fixed in space.

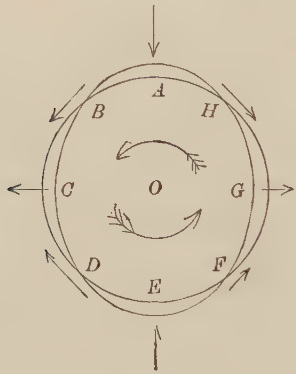


Fig. 1.

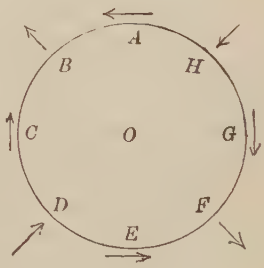


Fig. 2.

To show that the nodes do not rotate as if fixed in the ring, let the small arrows in Fig. 1 represent the directions of *relative* motion of the particles exclusive of the components due to rotation. At A, E, the particles are moving towards the centre O. This will of course increase their actual angular velocity and will give them a relative angular acceleration in the direction of rotation, as

represented by the arrows at A, E in Fig. 2. At C, G the particles are moving outwards, and this will retard their angular velocity. The particles at B, F are moving with greater total angular velocity than the rest; this will increase their "centrifugal force" and give them a relative acceleration outwards. Those at D, H are moving with the least total angular velocity, and the diminution in centrifugal force will give them a relative acceleration inwards. Hence the rotatory motion of the mass will give rise to relative accelerations of the particles in directions represented by the arrows in Fig. 2. If we compare the arrows in Fig. 1 and Fig. 2 we see at once that the effect of these relative accelerations is to cause retrograde motion of the nodes relative to the mass, that is, the nodes will rotate less rapidly than the ring. This explanation is obviously applicable to all the modes of vibration.

We will now determine the frequency-equations for the two-dimensional vibrations of a thin cylindrical shell or ring of radius a which is rotating about its axis with angular velocity ω*. We shall suppose that the cylinder is also acted on by an attractive force μ times the distance, directed towards the axis. The introduction of this attraction will enable us to separate the purely statical effects of centrifugal force, since by taking $\mu = \omega^2$ the latter effects will be counteracted. Unless this condition is satisfied, the circumference of the cylinder will be in a state of tension. Let this tension be T (per unit length of generator); and let σ be the surface density of the cylindrical shell or the line density of the ring.

When the cylinder is rotating steadily, the condition for relative equilibrium gives by resolving normally

$$\sigma\omega^2 a = \frac{T}{a} + \sigma\mu a,$$

therefore $$T = \sigma a^2 (\omega^2 - \mu) \quad \text{.....................(1)}.$$

In order to define the position of any point on the cylinder at any time t, it will be convenient to employ two systems of polar co-ordinates having the centre as pole, in one of which the initial line is fixed while in the other it revolves with angular velocity ω. If, in the undisturbed state the polar co-ordinates in the two systems are (a, ϕ) and (a, θ), we shall have

$$\phi = \theta + \omega t,$$

and θ will be constant for any particle of the ring.

In the small oscillations, let the small relative tangential and radial displacements of the particle be v and w so that its new polar co-ordinates are $(a + w,\ \phi + v/a)$ or $(a + w,\ \theta + v/a)$ in the two systems respectively.

* Compare Lord Rayleigh, *Theory of Sound*, I. p. 322.

As we shall require to apply the variational equation of energy, we must calculate $\delta\mathfrak{T}$ the virtual work of the effective forces of the system. Now if f_1, f_2 denote the transversal and radial accelerations of any point, the well-known formulæ applicable to polar co-ordinates give

$$\begin{aligned} f_1 &= \frac{1}{a+w}\frac{d}{dt}\{(a+w)^2(\omega+\dot{v}/a)\} \\ &= \ddot{v} + 2\omega\dot{w}, \\ f_2 &= \ddot{w} - (a+w)(\omega+\dot{v}/a)^2 \\ &= -a\omega^2 + \ddot{w} - 2\omega\dot{v} - w\omega^2 \end{aligned}$$

(neglecting squares and products of the small displacements v, w).

Hence

$$\begin{aligned} \delta\mathfrak{T} &= \int_0^{2\pi} (f_1\delta v + f_2\delta w)\,\sigma a d\theta \\ &= \sigma a \int_0^{2\pi} \{-a\omega^2\delta w + (\ddot{v} + 2\omega w)\,\delta v \\ &\qquad + (\ddot{w} - 2\omega\dot{v} - w\omega^2)\,\delta w\}\,\delta\theta \text{............(2).} \end{aligned}$$

The variation of potential energy will consist of three terms representing respectively the work done against the tension T in stretching the circumference, the work done against the attracting force, and that of bending the cylinder. We shall denote the potential energies due to these three causes by W_1, W_2 and V respectively, and their variations will be δW_1, δW_2 and δV.

To find δW_1, let e be the extension produced by the displacements (v, w) in the arc $ad\theta$. By writing down the stretched length of the arc we have

$$(1+e)^2(ad\theta)^2 = dw^2 + (a+w)^2(d\theta + dv/a)^2;$$

$$\therefore\ \ 2e + e^2 = \frac{2}{a}\left(w + \frac{dv}{d\theta}\right) + \frac{1}{a^2}\left(w + \frac{dv}{d\theta}\right)^2 + \frac{1}{a^2}\left\{\left(\frac{dw}{d\theta}\right)^2 + 2w\frac{dv}{d\theta}\right\},$$

and therefore, to the second order

$$e = \frac{1}{a}\left(w + \frac{dv}{d\theta}\right) + \frac{1}{2a^2}\left\{\left(\frac{dw}{d\theta}\right)^2 + 2w\frac{dv}{d\theta}\right\} \text{............(3).}$$

Hence

$$\begin{aligned} \delta W_1 &= \int_0^{2\pi} T\delta e\,.\,ad\theta \\ &= \int_0^{2\pi} T\left\{\delta w + \frac{d\delta v}{d\theta} + \frac{1}{a}\left(\frac{dw}{d\theta}\frac{d\delta w}{d\theta} + w\frac{d\delta v}{d\theta} + \frac{dv}{d\theta}\delta w\right)\right\} d\theta \\ &= \int_0^{2\pi} T\delta w d\theta + \int_0^{2\pi}\frac{T}{a}\left\{\left(\frac{dv}{d\theta} - \frac{d^2w}{d\theta^2}\right)\delta w - \frac{dw}{d\theta}\delta v\right\} d\theta \text{...(4),} \end{aligned}$$

by integrating by parts.

Also

$$W_2 = \int_0^{2\pi} \tfrac{1}{2}\mu\sigma \{(a+w)^2 - a^2\}\, ad\theta$$

$$= \int_0^{2\pi} \tfrac{1}{2}\mu\sigma a\,(2aw + w^2)\, d\theta;$$

$$\therefore \;\; \delta W_2 = \int_0^{2\pi} \mu\sigma a^2 \delta w d\theta + \int_0^{2\pi} \mu\sigma a w \delta w d\theta \quad\ldots\ldots\ldots\ldots(5).$$

Finally for the energy of bending we have if $\Delta(1/R)$ is the change of curvature,

$$V = \tfrac{1}{2}\beta \int_0^{2\pi} \left(\Delta \frac{1}{R}\right)^2 ad\theta = \tfrac{1}{2}\frac{\beta}{a^3}\int_0^{2\pi} \left(\frac{d^2w}{d\theta^2} + w\right)^2 d\theta,$$

where, for a cylindrical shell of thickness $2h$

$$\beta = \tfrac{2}{3}h^3 \left(\frac{E}{1-\mu'^2} + T\right),$$

and for a ring* $\qquad \beta = (E+T)\,I.$

We readily find $\qquad \delta V = \dfrac{\beta}{a^3}\displaystyle\int_0^{2\pi} \left(\frac{d^2}{d\theta^2} + 1\right)^2 w\delta w d\theta \quad\ldots\ldots\ldots\ldots(6).$

The variational equation of motion

$$\delta\mathfrak{T} + \delta W_1 + \delta W_2 + \delta V = 0$$

becomes, therefore, on slightly rearranging the terms,

$$0 = \int_0^{2\pi} \{\sigma a^2(-\omega^2 + \mu) + T\}\, \delta w d\theta$$

$$+ \int_0^{2\pi} \Big\{ \sigma a\,(\ddot{v} + 2\omega\dot{w})\,\delta v + \sigma a\,(\ddot{w} - 2\omega\dot{v} - \omega^2 w)\,\delta w$$

$$- \frac{T}{a}\frac{dw}{d\theta}\,\delta v + \left[\frac{T}{a}\left(\frac{dv}{d\theta} - \frac{d^2w}{d\theta^2}\right) + \mu\sigma a w + \frac{\beta}{a^3}\left(\frac{d^2}{d\theta^2} + 1\right)^2 w\right]\delta w \Big\}\, d\theta.$$

For the undisturbed motion we have $v = 0$, $w = 0$, and the first line of the above expression gives

$$T = \sigma a^2(\omega^2 - \mu),$$

as already found (1).

For the oscillations, we must assume with Lord Rayleigh that the extension vanishes to the first order, so that

$$\left.\begin{aligned} w &= -\frac{dv}{d\theta} \\ \text{and} \qquad \delta w &= -\frac{d\delta v}{d\theta} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(7).$$

* See Lord Rayleigh, *Theory of Sound*, I. p. 242. Here E denotes Young's modulus and μ' Poisson's ratio.

Substituting these expressions for w, δw and integrating by parts the terms containing $d\delta v/d\theta$, we find the equation of motion

$$\ddot{v} - \frac{d^2\ddot{v}}{d\theta^2} - 4\omega\frac{d\dot{v}}{d\theta} + (\omega^2 - \mu)\frac{d^2 v}{d\theta^2}$$
$$+ \frac{T}{\sigma a^2}\left(2\frac{d^2 v}{d\theta^2} + \frac{d^4 v}{d\theta^4}\right) - \frac{\beta}{\sigma a^4}\frac{d^2}{d\theta^2}\left(\frac{d^2}{d\theta^2} + 1\right)^2 v = 0.$$

To find the frequencies, assume that

$$v = A\cos(s\theta + pt)\text{.......................}(8).$$

Then the last equation gives (substituting for T its value by (1))

$$(1 + s^2)p^2 - 4\omega ps = (\omega^2 - \mu)s^2(s^2 - 3) + \frac{\beta}{\sigma a^4}s^2(s^2 - 1)^2\text{......}(9),$$

therefore

$$\left(p - \frac{2s\omega}{s^2 + 1}\right)^2 = \frac{4\omega^2 s^2}{(s^2 + 1)^2} + (\omega^2 - \mu)\frac{s^2(s^2 - 3)}{s^2 + 1} + \frac{\beta}{\sigma a^4}\frac{s^2(s^2 - 1)^2}{s^2 + 1}\text{...}(10).$$

Let $$\varpi_s^{\,2} = \frac{4\omega^2 s^2}{(s^2 + 1)^2} + (\omega^2 - \mu)\frac{s^2(s^2 - 3)}{s^2 + 1} + \frac{\beta}{\sigma a^4}\frac{s^2(s^2 - 1)^2}{s^2 + 1}\text{...}(11),$$

then the two values of p are p_1, p_2, where

$$p_1 = \frac{2s\omega}{s^2 + 1} + \varpi_s, \quad p_2 = \frac{2s\omega}{s^2 + 1} - \varpi_s,$$

and the corresponding motions of the small corrugations relatively to the mass are determined by

$$v = A\cos\left\{s\theta + \frac{2s\omega}{s^2 + 1}t + \varpi_s t\right\},$$

and $$v = A\cos\left\{s\theta + \frac{2s\omega}{s^2 + 1}t - \varpi_s t\right\}$$

respectively, together with the relation $w = -dv/d\theta$.

To find the actual motion in space, substitute $\phi - \omega t$ for θ in the two last equations; the positions of the corrugations will now be referred to the fixed initial line. We find for the two types of oscillation, respectively

$$v = A\cos\left\{s\phi - \frac{s^2 - 1}{s^2 + 1}s\omega t + \varpi_s t\right\},$$

and $$v = A\cos\left\{s\phi - \frac{s^2 - 1}{s^2 + 1}s\omega t - \varpi_s t\right\}.$$

If the amplitude (A) is the same in both, we see, by addition, that their resultant is given by

$$v = 2A\cos\varpi_s t\cos s\left(\phi - \frac{s^2 - 1}{s^2 + 1}\omega t\right)\text{...........}(12).$$

This may be interpreted as representing oscillations of the ring or cylinder of period $2\pi/\varpi_s$ having $2s$ nodes or nodal meridians (where $dv/d\phi = 0$) which rotate about the axis with angular velocity

$$\frac{s^2-1}{s^2+1}\,\omega.$$

The nodal angular velocity is thus in every case less than ω, but in the higher tones its difference from ω becomes less and less.

As the nodes are carried round in succession past the direction of the observer's ear, beats will be heard, and their number per revolution of the material will be

$$2s\,\frac{s^2-1}{s^2+1}.$$

The numerical values of these results are tabulated below for a few of the smaller values of s:

s	Number of Nodes. $2s$	Nodal ang. vel. / Ang. vel. of ring $\frac{s^2-1}{s^2+1}$	Number of beats per revolution. $2s\frac{s^2-1}{s^2+1}$
2	4	·6	2·4
3	6	·8	4·8
4	8	·882	7·005
5	10	·923	9·231
6	12	·946	11·351
7	14	·96	13·44

The pitch of the intermittent sound is determined by ϖ_s. Now when $\omega = 0$, $\mu = 0$, we have

$$\varpi_s^2 = \frac{\beta}{\sigma a^4}\,\frac{s^2(s^2-1)^2}{s^2+1} \quad\text{.......................}(13),$$

as found by Hoppe* and Lord Rayleigh†. Denoting this value of ϖ_s by Π_s, equation (11) shows that if $\mu = \omega^2$, so that the attraction counterbalances the purely statical effects of centrifugal force, we shall have $\varpi_s^2 > \Pi_s^2$, or the pitch will be somewhat raised by the rotation.

If however $\mu = 0$, and the frequency of rotation is small compared with the frequency of the vibrations in a non-rotating cylinder, so that ω is a small quantity of the first order compared with Π_s, equation (11) shows that ϖ_s differs from Π_s by small

* *Crelle*, Bd. 63, 1871. † *Theory of Sound.*

quantities of the second order. This shows that in all cases of practical interest the pitch is not perceptibly raised by the rotatory motion, and the only noticeable effect is that of the beats already described. On the other hand, if the cylinder is revolving very rapidly, so that ω is comparable with Π_s, the vibrations will no longer give the effect of beats at all*.

The results of the last paragraph are very important because they admit of extension to rotating shells in general, and afford us an easy way of determining the nodal angular velocity and consequent number of beats per revolution in other slowly-revolving systems where the vibrations are not two-dimensional. It will be seen that the nodal rotation depends exclusively on $\delta\mathfrak{T}$ the variation of the kinetic energy, and that the square of ω may be neglected throughout, since it only appears in the expression for the frequency, and does not affect the latter to any appreciable extent. All that is necessary, therefore, is to calculate $\delta\mathfrak{T}$. We proceed to apply this method to the vibrations which Lord Rayleigh has investigated for a perfectly inextensible cylindrical shell of length l closed by an inextensible disk at one end†.

Taking the axis of the cylinder as axis of z, and the closed end at $z=0$, we have, if u denote the longitudinal displacement,

$$\delta\mathfrak{T}=\int_0^l dz\int_0^{2\pi}\{\ddot{u}\delta u+(\ddot{v}+2\omega\dot{w})\,\delta v+(\ddot{w}-2\omega\dot{v})\,\delta w\}\,\sigma a d\theta\;\ldots(14),$$

omitting the terms $-a\omega^2\delta w-w\omega^2\delta w$; of which the first depends on the undisturbed motion, while the second involves small quantities of the second order. Neither of these omitted terms will in any case affect the nodal rotation as they do not contain differential coefficients with regard to the time.

Lord Rayleigh's conditions of inextensibility are

$$\frac{du}{dz}=0,\quad w+\frac{dv}{d\theta}=0,\quad \frac{du}{d\theta}+a\frac{dv}{dz}=0\;\ldots\ldots\ldots\ldots(15),$$

and we suppose δu, δv, δw to satisfy similar conditions. By means of the second of these conditions we eliminate w, δw and have, by integrating by parts,

$$\delta\mathfrak{T}=\int_0^l dz\int_0^{2\pi}\left\{\ddot{u}\delta u+\left(\ddot{v}-\frac{d^2\ddot{v}}{d\theta^2}-4\omega\frac{d\dot{v}}{d\theta}\right)\delta v\right\}\sigma a d\theta\ldots\ldots(16).$$

From the first and third we see that u is a function of θ and not of z, and therefore, that

$$v=-\frac{z}{a}\frac{du}{d\theta}.$$

* By putting $\beta=0$, $\mu=0$ in (10) we may deduce the solution to the purely kinetic problem of determining the oscillations of a rapidly revolving flexible endless chain.

† *Proc. London Mathematical Soc.* XIII., page 5. *Proceedings Royal Society*, Vol. XLV.

This enables us to eliminate v, δv, and therefore by integrating by parts we find

$$\delta\mathfrak{T} = \int_0^l dz \int_0^{2\pi} \left\{ \ddot{u} - \frac{z^2}{a^2} \frac{d^2}{d\theta^2} \left(\ddot{u} - \frac{d^2\ddot{u}}{d\theta^2} - 4\omega \frac{d\dot{u}}{d\theta} \right) \right\} \delta u \sigma a d\theta.$$

We have not yet assumed the density σ to be independent of z. Making this assumption and integrating with respect to z we have

$$\delta\mathfrak{T} = \int_0^{2\pi} \left\{ u - \tfrac{1}{3} \frac{l^2}{a^2} \frac{d^2}{d\theta^2} \left(\ddot{u} - \frac{d^2\ddot{u}}{d\theta^2} - 4\omega \frac{d\dot{u}}{d\theta} \right) \right\} \delta u \sigma l a d\theta \ldots (17),$$

and this shows that the differential equation for u may be written in the form

$$\ddot{u} - \tfrac{1}{3} \frac{l^2}{a^2} \frac{d^2}{d\theta^2} \left(\ddot{u} - \frac{d^2\ddot{u}}{d\theta^2} - 4\omega \frac{d\dot{u}}{d\theta} \right) = F\left(\frac{d^2}{d\theta^2} \right)(u),$$

the right-hand side containing no differential coefficients with regard to the time. Taking u proportional to $\cos(s\phi + pt)$ we have

$$p^2 + \tfrac{1}{3} \frac{l^2}{a^2} s^2 (p^2 + s^2p^2 - 4\omega sp) = F(-s^2).$$

This may be put in the form

$$\left\{ p - \frac{2s\omega}{s^2 + 1 + \dfrac{3a^2}{l^2s^2}} \right\}^2 = \varpi_s^2 \ldots\ldots\ldots\ldots\ldots\ldots (18),$$

where ϖ_s is a function of s. Comparing this form with (10), the corresponding form for the two dimensional oscillations, it is easy to see that, in the present case, the nodal rate of revolution will be

$$\frac{s^2 + \dfrac{3a^2}{l^2s^2} - 1}{s^2 + \dfrac{3a^2}{l^2s^2} + 1} \omega,$$

and the number of beats per revolution will be

$$2s \cdot \frac{s^2 + \dfrac{3a^2}{l^2s^2} - 1}{s^2 + \dfrac{3a^2}{l^2s^2} + 1}.$$

But we may generalise still further. In any surface of revolution, one of Lord Rayleigh's conditions of inextensibility is

$$w + \frac{dv}{d\theta} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19).$$

If we form $\delta\mathfrak{T}$, using this and the other two conditions, it is evident that we shall arrive at an equation for p of the form

$$\lambda_s p^2 + (1 + s^2) p^2 - 4\omega sp = \text{a function of } s \ldots\ldots (20),$$

where the term $\lambda_s p^2$ is derived from the terms $\ddot{u}\delta u$ which represent the virtual work of the effective forces due to the longitudinal components (u) of the displacement, and it is important to notice that λ_s *can never be negative.* Although this last statement is not obvious from the *variational* equation required for the treatment of a *revolving* shell, it becomes evident from the consideration that in a *non-revolving* shell the variational equation leads to the same equation for p as the principle of conservation of energy, and that in the equation of energy the term containing $\lambda_s p^2$ will arise from the kinetic energy of the longitudinal motion ($\dot{u}$). This kinetic energy is of course essentially positive; or, in other words, the whole kinetic energy of the system is greater than it would be if the longitudinal motion were neglected. Hence λ_s is positive.

The nodal angular velocity,

$$\frac{s^2+\lambda_s-1}{s^2+\lambda_s+1}\,\omega,$$

and the number of beats per revolution,

$$2s\,\frac{s^2+\lambda_s-1}{s^2+\lambda_s+1},$$

are therefore both greater than they would be if there were no longitudinal motion.

The limiting case is that of a plane circular plate revolving about an axis perpendicular to its plane. Here v, w are both zero, and the nodal radii are fixed relatively to the revolving mass, the vibrations being unaffected by the rotation. The nodal angular velocity is therefore ω, and the number of beats per revolution is $2s$.

Now in a communication read before the British Association at Leeds, I announced that experiments with two different champagne glasses attached to a microscopist's turn-table gave about 2·6 and 2·2 beats per revolution respectively for the gravest tone. While there is nothing contradictory in the former result, the latter is too small to be compatible with our theory. As the numbers were found by counting the beats during about eight revolutions of the table and the mean of 26 observations was taken, it is impossible that the discrepancy can arise wholly from errors of observation. A further possible source of error was the want of uniformity in the angular velocity of the glass. As a matter of fact, however, the beats seemed, if anything, most rapid when the glass was first set in motion, and as it was not brought to rest again during the interval in which the beats were counted, I rather doubt this as the cause of the difference. On the whole I should be rather inclined to favour the idea that the discrepancy

is due to Lord Rayleigh's conditions of inextensibility not being strictly fulfilled in the neighbourhood of the free edge.

The results of this paper may therefore be of interest in connection with the recent controversy on this subject by showing how far Lord Rayleigh's theory of thin shells is capable of *practical* application. We see that, if his conditions of inextensibility hold good, the number of beats per revolution depends only on the shape of the surface and on the law of distribution of density, and is in no way dependent on the elasticity of the substance. It is therefore readily calculable for a given thin bell. Moreover, the number of beats per revolution when the bell is rotated uniformly, can easily be counted. If there should be any discrepancy between the observed and calculated results which is otherwise unaccounted for, this will give us a probable indication to what extent the deformation differs from one of pure bending in the bell which is the subject of our experiments.

(2) *On Liquid Jets* (continued). By H. J. SHARPE, M.A., St John's College.

1. In Vol. VII. Pt. I. I gave an approximate solution of a case of liquid flowing from a vessel and becoming a jet, the ultimate breadth of the jet being half the diameter of the vessel. I now propose to give in detail another case which may perhaps have more interest, since in the following the diameter of the orifice may be as small as we please compared with the diameter of the vessel. In the former solution some ambiguity was attached to the position of the orifice. In the present solution there is none. Some further observations will be made after the solution has been obtained.

2. We take a case where the outer stream-line (fig. 1) $AFGBHC$ cuts the axis of y in a point B such that $OB = Cx'$ the semi-breadth of the jet at infinity.

Let ψ be the stream function on the *left* of Oy.

Let $OE = \pi$, $OB = Cx' = \pi/p$, where p is supposed to be a *large integer*.

On the *left* of Oy, let

$$\left.\begin{aligned} -\frac{d\psi}{dy} \equiv -u = a_1\epsilon^{x}\cos y + a_3\epsilon^{3x}\cos 3y + a_5\epsilon^{5x}\cos 5y \\ + \Sigma c_n\epsilon^{pnx}\cos pny + A \\ -\frac{d\psi}{dx} \equiv v = a_1\epsilon^{x}\sin y + a_3\epsilon^{3x}\sin 3y + a_5\epsilon^{5x}\sin 5y \\ + \Sigma c_n\epsilon^{pnx}\sin pny \end{aligned}\right\} \quad \text{.........} (1),$$

where a_1, a_3, a_5, and c_n are arbitrary constants, and Σ means summation with regard to n from 1 to ∞. (See Art. 9.)

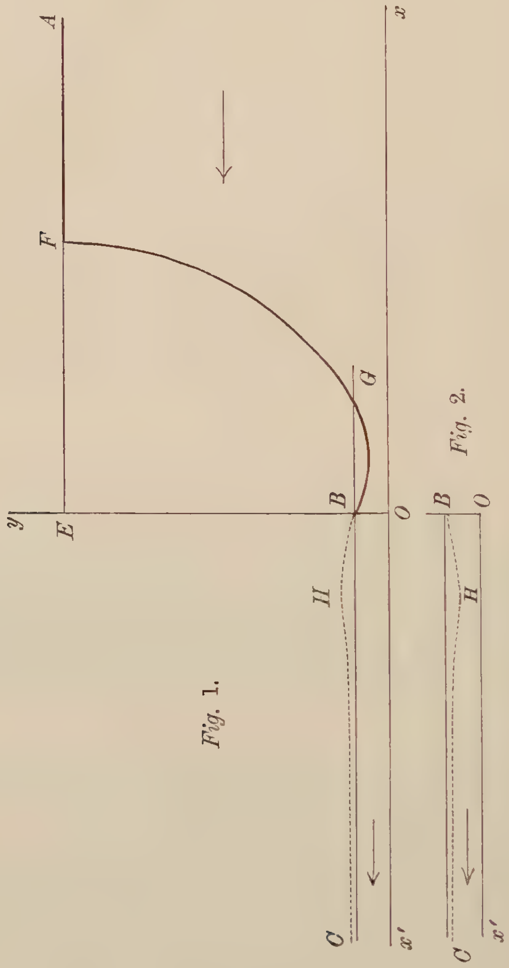


Fig. 1.

Fig. 2.

Then the equation to BHC is

$$a_1\epsilon^{x}\sin y + \tfrac{1}{3}a_3\epsilon^{3x}\sin 3y + \tfrac{1}{5}a_5\epsilon^{5x}\sin 5y + \Sigma\frac{c_n}{pn}\epsilon^{pnx}\sin pny + Ay$$

$$= a_1\sin\frac{\pi}{p} + \tfrac{1}{3}a_3\sin\frac{3\pi}{p} + \tfrac{1}{5}a_5\sin\frac{5\pi}{p} + \frac{A\pi}{p}\ldots(2).$$

Since $Cx' = \pi/p$ we must have

$$a_1 \sin\frac{\pi}{p} + \tfrac{1}{3}a_3 \sin\frac{3\pi}{p} + \tfrac{1}{5}a_5 \sin\frac{5\pi}{p} = 0 \quad\text{.............(3).}$$

It will be found convenient to replace a_1, a_3, a_5 by other quantities such that

$$a_1 \equiv \alpha_1 + A_1, \quad a_3 \equiv \alpha_3 + A_3, \quad a_5 \equiv \alpha_5 + A_5 \quad\text{.........(4).}$$

When $x = 0$, we have along OB, on the *left* of it,

$$\left.\begin{aligned} -u &= (\alpha_1 + A_1)\cos y + (\alpha_3 + A_3)\cos 3y + (\alpha_5 + A_5)\cos 5y \\ &\qquad + \Sigma c_n \cos pny + A \\ v &= (\alpha_1 + A_1)\sin y + (\alpha_3 + A_3)\sin 3y + (\alpha_5 + A_5)\sin 5y \\ &\qquad + \Sigma c_n \sin pny \end{aligned}\right\} \quad\text{.... (5).}$$

Let χ be the stream function on the *right* of Oy, and on the *right* of Oy let

$$\left.\begin{aligned} -\frac{d\chi}{dy} &\equiv -u = b_1\epsilon^{-x}\cos y + b_3\epsilon^{-3x}\cos 3y + b_5\epsilon^{-5x}\cos 5y \\ &\qquad + \Sigma c_n'\epsilon^{-pnx}\cos pny + B \\ -\frac{d\chi}{dx} &\equiv v = -b_1\epsilon^{-x}\sin y - b_3\epsilon^{-3x}\sin 3y - b_5\epsilon^{-5x}\sin 5y \\ &\qquad - \Sigma c_n'\epsilon^{-pnx}\sin pny \end{aligned}\right\} \quad\text{........ (6).}$$

It will be found convenient to put

$$b_1 \equiv \alpha_1 - A_1, \quad b_3 \equiv \alpha_3 - A_3, \quad b_5 \equiv \alpha_5 - A_5 \quad\text{.........(7).}$$

When $x = 0$, we have along OB, on the *right* of it,

$$\left.\begin{aligned} -u &= (\alpha_1 - A_1)\cos y + (\alpha_3 - A_3)\cos 3y + (\alpha_5 - A_5)\cos 5y \\ &\qquad + \Sigma c_n'\cos pny + B \\ v &= -(\alpha_1 - A_1)\sin y - (\alpha_3 - A_3)\sin 3y - (\alpha_5 - A_5)\sin 5y \\ &\qquad - \Sigma c_n'\sin pny \end{aligned}\right\} \quad\text{.... (8).}$$

By Fourier's Theorem, suppose we have from $y = 0$ to π/p

$$2A_1\cos y + 2A_3\cos 3y + 2A_5\cos 5y = Q + \Sigma q_n \cos pny \quad\text{...(9),}$$

which will be true at both limits.

Identifying the first of (5) and the first of (8) we have

$$c_n - c_n' + q_n = 0 \quad\text{........................(10),}$$

$$A - B + Q = 0 \quad\text{........................(11).}$$

Again, suppose we have from $y = 0$ to π/p

$$2\alpha_1\sin y + 2\alpha_3\sin 3y + 2\alpha_5\sin 5y = \Sigma r_n \sin pny \quad\text{... (12),}$$

which will be true at both limits if

$$\alpha_1 \sin\frac{\pi}{p} + \alpha_3 \sin\frac{3\pi}{p} + \alpha_5 \sin\frac{5\pi}{p} = 0 \quad\ldots\ldots\ldots\ldots(13).$$

Identifying the second of (5) and the second of (8) we have

$$c_n + c_n' + r_n = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

From (6) the equation to $AFGB$ is

$$b_1\epsilon^{-x}\sin y + \tfrac{1}{3}b_3\epsilon^{-3x}\sin 3y + \tfrac{1}{5}\epsilon^{-5x}\sin 5y + \Sigma\frac{c_n'}{pn}\epsilon^{-pnx}\sin pny + By$$
$$= b_1\sin\frac{\pi}{p} + \tfrac{1}{3}b_3\sin\frac{3\pi}{p} + \tfrac{1}{5}b_5\sin\frac{5\pi}{p} + \frac{B\pi}{p} \quad\ldots\ldots (15).$$

Since $Ax = \pi$, we must have

$$b_1\sin\frac{\pi}{p} + \tfrac{1}{3}b_3\sin\frac{3\pi}{p} + \tfrac{1}{5}b_5\sin\frac{5\pi}{p} = \frac{(p-1)B\pi}{p} \quad\ldots (16).$$

It has already been shewn that there is a sharp turn at F.

From (9) we have

$$Q = 2A_1\frac{p}{\pi}\sin\frac{\pi}{p} + 2A_3\frac{p}{3\pi}\sin\frac{3\pi}{p} + 2A_5\frac{p}{5\pi}\sin\frac{5\pi}{p} \quad\ldots(17).$$

$$q_n = -4A_1\frac{p}{\pi}\sin\frac{\pi}{p}\times\frac{\cos n\pi}{p^2n^2-1} - 4A_3\frac{3p}{\pi}\sin\frac{3\pi}{p}\times\frac{\cos n\pi}{p^2n^2-9}$$
$$-4A_5\frac{5p}{\pi}\times\sin\frac{5\pi}{p}\times\frac{\cos n\pi}{p^2n^2-25} \quad\ldots\ldots(18).$$

From (12) we have

$$r_n = -4\alpha_1\frac{p^2}{\pi}\sin\frac{\pi}{p}\times\frac{n\cos n\pi}{p^2n^2-1} - 4\alpha_3\frac{p^2}{\pi}\sin\frac{3\pi}{p}\times\frac{n\cos n\pi}{p^2n^2-9}$$
$$-4\alpha_5\frac{p^2}{\pi}\sin\frac{5\pi}{p}\times\frac{n\cos n\pi}{p^2n^2-25} \quad\ldots\ldots(19).$$

From (10) and (14)
$$\left.\begin{aligned} c_n &= -\tfrac{1}{2}r_n - \tfrac{1}{2}q_n \\ c_n' &= -\tfrac{1}{2}r_n + \tfrac{1}{2}q_n \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots(20).$$

When the above conditions are satisfied, the velocities on each side of OB will be continuous.

3. Equations (3), (13) are the only equations connecting the 6 quantities α_1, A_1, α_3, A_3, α_5, A_5, therefore so far 4 are independent. Equations (11), (16), (17) serve to connect A with B and either with α_1, A_1, &c. From (20) c_n, c_n' are functions of n and α_1, A_1, &c. The 6 quantities α_1, A_1, &c. may any of them (consistently with the above relations) be as small as we like, but none of them

will be taken so large as to be comparable with p. We will suppose p so large that

$$\frac{p}{\pi}\sin\frac{\pi}{p},\quad \frac{p}{3\pi}\sin\frac{3\pi}{p},\quad \frac{p}{5\pi}\sin\frac{5\pi}{p}$$

are very nearly unity.

For some investigations connected with this subject it might be necessary to expand these expressions in powers of $1/p$, but for our present purpose it will suffice if we suppose them actually unity. To this end we must suppose p to be at least 22 (this number making the largest rejected term $\frac{1}{10}$th of what is retained), but we may have to take p much larger than this.

(3) and (13) then become

$$a_1 + a_3 + a_5 = 0 \quad\text{.......................(21)},$$

$$\alpha_1 + 3\alpha_3 + 5\alpha_5 = 0 \quad\text{.......................(22)}.$$

(16) and (17) become

$$b_1 + b_3 + b_5 = pB \quad\text{.......................(23)},$$

$$Q = 2A_1 + 2A_3 + 2A_5 \quad\text{.......................(24)}.$$

From (11), (21), (23), (24) we readily get

$$A = pB = -Q = -2A_1 - 2A_3 - 2A_5 \quad\text{........(25)}.$$

We will next suppose p so large that we may safely expand the fractions in (18) and (19) in ascending powers of $1/p$. To this end p must be at least 50, and we shall get, retaining only the most important terms in q_n and r_n,

$$q_n = -\frac{4\cos n\pi}{p^2n^2}(A_1 + 9A_3 + 25A_5) \quad\text{...........(26)}.$$

$$r_n = -\frac{4\cos n\pi}{p^3n^3}(\alpha_1 + 27\alpha_3 + 125\alpha_5) \quad\text{...........(27)}.$$

For large values of p, r_n vanishes compared with q_n. We shall therefore retain only q_n in equations (20). We shall further suppose a_1, a_3, a_5 to be small quantities multiples of $1/p^2$ and that their ratios are such that

$$a_1 + 3a_3 + 5a_5 = 0 \quad\text{.....................(28)}.$$

(The object of these assumptions will be presently seen.)

From (21) and (28) therefore we may put

$$-a_1 = \tfrac{1}{2}a_3 = -a_5 = p_2 \quad\text{.....................(29)},$$

where p_2 is a small quantity supposed to be a multiple of $1/p^2$.

It will then be found from equations (4), (22) and (29), that so far we may regard α_1 and α_3 as arbitrary, and A_1, A_3, A_5, α_5 as determined in terms of them by means of the following equations:

$$\left.\begin{array}{r} A_1 = -\alpha_1 - p_2,\ A_3 = -\alpha_3 + 2p_2,\ A_5 = \frac{1}{5}(\alpha_1 + 3\alpha_3) - p_2 \\ \alpha_5 = -\frac{1}{5}(\alpha_1 + 3\alpha_3) \end{array}\right\}\ldots(30).$$

We shall further get, omitting a small term multiplied by p_2 in the bracket following,

$$c_n = -c_n' = \frac{4 \cos n\pi}{p^2 n^2}(2\alpha_1 + 3\alpha_3) \ldots\ldots\ldots\ldots(31),$$

$$A = pB = \tfrac{8}{5}\alpha_1 + \tfrac{4}{5}\alpha_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32).$$

4. We shall now make BHC as far as possible a line of constant velocity. In (2), putting for shortness z for ϵ^x and modifying by means of (29) and (31), we get for the equation to BHC

$$A\left(y - \frac{\pi}{p}\right) = p_2(z \sin y - \tfrac{2}{3}z^3 \sin 3y + z^5 \sin 5y)$$
$$- \frac{4}{p^3}(2\alpha_1 + 3\alpha_3)\,\Sigma \frac{\cos n\pi}{n^3} z^{pn} \sin pny \ldots(33).$$

It is obvious that for any point on BHC the coefficient of p_2 cannot exceed $\frac{4}{3}$ and that the Σ cannot exceed $\Sigma(1/n^3)$ which is about 1·18. All along BHC therefore y differs from π/p only by quantities at most of the 2nd order of smallness. (They are in fact of the 3rd order.) In (1) therefore we may put unity for $\cos y$, $\cos 3y$ and $\cos 5y$, and then modifying (1) by means of (29) and (31), we have at every point on BHC

$$-u = p_2(-z + 2z^3 - z^5)$$
$$+ \frac{4}{p^2}(2\alpha_1 + 3\alpha_3)\,\Sigma \frac{\cos n\pi}{n^2} z^{pn} \cos pny + A \ldots(34).$$

From (1) and (31) and reasoning like the preceding, it is evident that v is at most of the 2nd order of smallness, because c_n is. Therefore $(u^2 + v^2)$ only differs from A^2 by quantities of the order $1/p^2$. Practically therefore at every point on BHC (34) gives the whole velocity. This velocity consists of a constant part A and a variable part. A portion of this variable part can be made, if we like, to vanish by putting

$$2\alpha_1 + 3\alpha_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35).$$

The remaining portion will be a maximum or minimum at a point determined by

$$-z + 6z^3 - 5z^5 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(36).$$

This is satisfied by $z = 1$ or the point B.

If p_2 is positive, we readily see that a *maximum* has been obtained.

From (1) and (28) v vanishes at B. From (33) we see that BHC is always above its asymptote. When $z^2 = \frac{1}{5}$ (36) furnishes a *minimum*, which, judging from the equation of continuity, should give us a maximum ordinate, which we indicate by H in the figures. This idea is corroborated, as we can prove from (1) that $z^2 = \frac{1}{5}$ causes v to vanish.

If p_2 is negative, the words "maximum and minimum" must be interchanged in the above sentences. BHC is then always below its asymptote, and at H there is a minimum ordinate or Vena Contracta (fig. 2).

In both cases, as we are not so much concerned with the sign, as with the actual magnitude of the error in the velocity derived from (34) we see, since the coefficient of p_2 in (34) vanishes when $z=1$, that the error is of the 3rd order of smallness at B and of the 2nd order at H, as far as p is concerned.

We now proceed to trace the curve $AFGB$ on the right of Oy.

5. From (7), (30) and (35) we get

$$\left.\begin{array}{l} b_1 = 2\alpha_1 + p_2,\ b_3 = -\frac{4}{3}\alpha_1 - 2p_2,\ b_5 = \frac{2}{5}\alpha_1 + p_2 \\ pB = \frac{16}{15}\alpha_1 \end{array}\right\} \text{......(37)},$$

also from (32),

therefore from (15) the equation to $AFGB$ becomes, putting z for ϵ^{-x},

$$\frac{16\alpha_1}{15p}(y-\pi) = -(2\alpha_1 + p_2)\, z \sin y$$
$$+ \left(\tfrac{4}{9}\alpha_1 + \tfrac{2}{3}p_2\right) z^3 \sin 3y - \left(\tfrac{2}{25}\alpha_1 + \tfrac{1}{5}p_2\right) z^5 \sin 5y \text{...(38)}.$$

As p_2 is small compared with α_1 we see that it makes little difference whether we put $+p_2$ or $-p_2$ in the last equation, as the curves obtained will only differ slightly in shape and position. In fact, to get a general idea of the form of the curve, we may of course practically put $p_2 = 0$.

We thus get for the abscissa of G where the curve cuts the line $y = \pi/p$

$$-\tfrac{16}{15} = -2z + \tfrac{4}{3}z^3 - \tfrac{2}{5}z^5,$$

whence $z = \cdot 628$ if x is about $\frac{1}{2}$.

At F $u = 0$, so we get from (6) for the abscissa of F

$$0 = -z + \tfrac{2}{3}z^3 - \tfrac{1}{5}z^5 + \frac{8}{15p},$$

EF therefore increases slowly as p increases (see Art. 8). If for instance $p = 100$, $EF = 5\frac{1}{3}$ nearly.

To consider more carefully the form of the small portion GB. As in this portion $y < \pi/p$ we may put in (38) y, $3y$, $5y$ for $\sin y$, $\sin 3y$, $\sin 5y$. We can then readily shew that in passing from the curve given by $+p_2$ to the curve given by $-p_2$ we alter the ordinate of GB by a quantity at most of the 3rd order of smallness*. It is interesting to compare with this the fact that the corresponding alteration in the ordinate of the *jet* at all points (except in the neighbourhood of B) is of the 2nd order of smallness. Thus a very small alteration in the vessel produces a disproportionately large alteration in the jet, in fact changing a maximum into a minimum ordinate or vice versâ.

The Problem thus seems to suggest a point of contact with Lord Rayleigh's article "On the Instability of Jets"—given in Vol. x. of the *Proceedings of the London Mathematical Society*, though it must be admitted that in that article the author is considering not the effect of the vessel on the form of the jet, but the instability of the jet itself due to capillary force.

6. Of course there is nothing whatever to prevent us putting $p_2 = 0$ in Art. 4, in which case we see from (34) that not only at B, but all along the jet the error in the velocity would be at most of the 3rd order of smallness.

We may remark that no attempt has been made to draw the figures to scale, as that would be difficult.

7. Supposing p_2 not to be zero, and the linear unit of measurement to be comparable with 1 foot (or perhaps better say $2'$ or $3'$), it would approximately make no difference to the solution if a

* From (38) we have in curved portion GB (putting a for a_1 for shortness),

$$\frac{16a}{15p}(y-\pi) = -(2a+p_2)zy + \left(\frac{4}{3}a+2p_2\right)z^3y - \left(\frac{2}{5}a+p_2\right)z^5y.$$

Now keeping z constant, suppose we put $-p_2$ for $+p_2$ and let y become y_1 then

$$\frac{16a}{15p}(y_1-\pi) = -(2a-p_2)zy_1 + \left(\frac{4}{3}a-2p_2\right)z^3y_1 - \left(\frac{2}{5}a-p_2\right)z^5y_1;$$

$\therefore$ subtracting, we have nearly

$$\frac{16a}{15p}(y-y_1) = (y-y_1)\left[-2az+\frac{4}{3}az^3-\frac{2}{5}az^5\right] + 2y\left[-p_2z+2p_2z^3-p_2z^5\right].$$

As in GB $\qquad z < \cdot 628,$

we have approximately

$$(y-y_1)\left[-2a\times\cdot 628 + \&c.\right] = -2yp_2\times\cdot 628;$$

$\therefore$ as in GB $\qquad y < \dfrac{\pi}{p}.$

$y-y_1$ is at *most* of the order $\qquad \dfrac{1}{p^3}.$

constant accelerating force g were acting on the fluid parallel to yO, for we see from (33) and (34) that in that case the equation $u^2+2gy=$ constant would be satisfied accurately to the 3rd order of small quantities all along BHC, as the coefficient of p_2 in (33) is of the order $1/p$. We thus get a sort of millrace BHC, and if p be large enough, AF might almost be regarded as a free surface.

8. It was proved in Art. 5, in the solutions already obtained, that EF is a function of p which increases slowly as p increases. Suppose it were required to find a solution wherein EF is a constant quantity independent of p, *but not zero*. We could do so by introducing into equations (1), (6) &c. additional terms involving (say) $\sin 7y$ and $\cos 7y$ with 2 additional arbitrary constants. The whole of the above process would then have to be gone through and it would be found quite possible to satisfy the new conditions. p would then have to be >70. The *order* of the errors would of course remain unchanged. In this case there is no point G, but the outer stream-line, after touching the asymptote at B, goes up to F. The figure on the right of Oy would then have a much closer resemblance to a cistern with an orifice at the bottom or (as we may suppose the figure symmetrical with regard to the axis of x) in the middle. There would be found to be an infinite number of such solutions. It must however be observed that *we cannot find a solution of this kind wherein EF is actually zero.* To prove this we will first consider the case treated in Arts. 1—6 above. Since from (31) and (35) $c_n'=0$, we have from (6) to determine the abscissa of F,

$$0=-b_1z-b_3z^3-b_5z^5+B.$$

If this is satisfied by $z=1$ we must have

$$0=-b_1-b_3-b_5+B \qquad (39).$$

But from (16), for large values of p,

$$b_1+b_3+b_5=B\pi \qquad (40).$$

(39) and (40) being incompatible, EF cannot be zero. An exactly similar proof would apply if we introduced into (1), (6) &c. additional terms involving (say) $\sin 7y$, $\cos 7y$.

9. In equations (1) (6) &c. the first three terms on the right-hand side involve the three odd numbers 1, 3, 5. We are not obliged to use odd numbers. Any integers will give a distinct case, provided that p is large compared with the largest of them. If we want a case which is compatible with the lowest possible value of p, we should choose the numbers 1, 2, 3, and then p would be as small as 30. So in Art. 8 if we chose the numbers 1, 2, 3, 4 we could have p as small as 40 and yet have EF to a considerable extent independent of p.

(3) *Note on the Application of Quaternions to the Discussion of Laplace's Equation.* By J. BRILL, M.A., St John's College.

1. The object of the following communication is to obtain, as far as is possible, with the aid of the Calculus of Quaternions, a theory for the three-dimensional form of Laplace's Equation analogous to the well-known theory of Conjugate Functions, which has proved of so much service in the treatment of the two-dimensional form.

The two related solutions of the two-dimensional theory are replaced in the three-dimensional theory, not by three, but by four related solutions. Thus if α, β, γ, δ be four quantities connected by the equations

$$\frac{\partial\delta}{\partial x} = \frac{\partial\gamma}{\partial y} - \frac{\partial\beta}{\partial z},$$

$$\frac{\partial\delta}{\partial y} = \frac{\partial\alpha}{\partial z} - \frac{\partial\gamma}{\partial x},$$

$$\frac{\partial\delta}{\partial z} = \frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y},$$

$$\frac{\partial\alpha}{\partial x} + \frac{\partial\beta}{\partial y} + \frac{\partial\gamma}{\partial z} = 0;$$

then it is easily verified that α, β, γ, δ are severally solutions of Laplace's Equation. Moreover, if we interpret δ as the velocity potential of a case of irrotational fluid motion, then α, β, γ are the components of the vector potential which occupies a similar place in the three-dimensional theory to that occupied by the current function in the two-dimensional theory.

If we now write

$$r = -\delta + i\alpha + j\beta + k\gamma,$$

we have

$$\nabla r = -\left(\frac{\partial\alpha}{\partial x} + \frac{\partial\beta}{\partial y} + \frac{\partial\gamma}{\partial z}\right) + i\left(-\frac{\partial\delta}{\partial x} + \frac{\partial\gamma}{\partial y} - \frac{\partial\beta}{\partial z}\right)$$
$$+ j\left(-\frac{\partial\delta}{\partial y} + \frac{\partial\alpha}{\partial z} - \frac{\partial\gamma}{\partial x}\right) + k\left(-\frac{\partial\delta}{\partial z} + \frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}\right) = 0 \ldots\ldots(1).$$

Further, let

$$\rho = -2ix + jy + kz,$$
$$\sigma = ix - 2jy + kz,$$
$$\tau = ix + jy - 2kz.$$

Then we have

$$\nabla\rho = \nabla\sigma = \nabla\tau = 0,$$

and therefore ρ, σ, τ are vector solutions of equation (1); but, since

$$\rho + \sigma + \tau = 0,$$

they only supply us with two independent solutions. This, however, will be sufficient for our purposes, since for the development of the theory we only require to know two independent special solutions. There is a certain disadvantage about these forms as the selection of two of them renders the work unsymmetrical. There is a symmetrical quaternion solution of (1), which involves x, y, z linearly, viz.

$$2(x+y+z) + i(y-z) + j(z-x) + k(x-y);$$

but I have not been able to hit upon a second symmetrical solution involving x, y, z linearly.

2. We are now in a position to shew that there exists a relation of the form

$$dr = d\rho \,.\, R + d\sigma \,.\, S \qquad\qquad (2),$$

where R and S are quaternions whose expressions involve x, y, z, but not dx, dy, dz. To verify this, assume

$$R = u + if + jg + kh,$$
$$S = v + il + jm + kn;$$

and then equation (2) becomes

$$-d\delta + id\alpha + jd\beta + kd\gamma = (-2idx + jdy + kdz)(u + if + jg + kh)$$
$$+ (idx - 2jdy + kdz)(v + il + jm + kn).$$

This involves the existence of the four relations

$$-d\delta = (2f - l)\,dx + (2m - g)\,dy - (h + n)\,dz,$$
$$d\alpha = (v - 2u)\,dx + (h - 2n)\,dy - (g + m)\,dz,$$
$$d\beta = (2h - n)\,dx + (u - 2v)\,dy + (f + l)\,dz,$$
$$d\gamma = (m - 2g)\,dx + (2l - f)\,dy + (u + v)\,dz;$$

and since these are to be satisfied independently of the values of the ratios $dx : dy : dz$, we obtain the following twelve equations:

$$\frac{\partial\delta}{\partial x} = l - 2f, \quad \frac{\partial\delta}{\partial y} = g - 2m, \quad \frac{\partial\delta}{\partial z} = h + n,$$

$$\frac{\partial\alpha}{\partial x} = v - 2u, \quad \frac{\partial\alpha}{\partial y} = h - 2n, \quad \frac{\partial\alpha}{\partial z} = -g - m,$$

$$\frac{\partial\beta}{\partial x} = 2h - n, \quad \frac{\partial\beta}{\partial y} = u - 2v, \quad \frac{\partial\beta}{\partial z} = f + l,$$

$$\frac{\partial\gamma}{\partial x} = m - 2g, \quad \frac{\partial\gamma}{\partial y} = 2l - f, \quad \frac{\partial\gamma}{\partial z} = u + v.$$

As we have here twelve equations, and only eight quantities to be determined, it is obvious that the equations imply four relations between the differential coefficients of α, β, γ, δ. It is to be remarked that the group of twelve equations consists of four sets containing three equations each, the three equations of any one set containing only a single pair of the above-mentioned eight quantities. The pairs are as follows: u and v, f and l, g and m, h and n. Thus each set of three equations will furnish us with a single relation, and enable us to determine, subject to that relation, the values of a pair of the eight quantities required.

The four relations connecting the differential coefficients of α, β, γ, δ are easily shewn to be identical with those contained in Article 1, and consequently the existence of equation (2) is justified. Thus we have:

$$\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z}=2l-f-f-l=l-2f=\frac{\partial\delta}{\partial x},$$

$$\frac{\partial\alpha}{\partial z}-\frac{\partial\gamma}{\partial x}=-g-m-m+2g=g-2m=\frac{\partial\delta}{\partial y},$$

$$\frac{\partial\beta}{\partial x}-\frac{\partial\alpha}{\partial y}=2h-n-h+2n=h+n=\frac{\partial\delta}{\partial z},$$

$$\frac{\partial\alpha}{\partial x}+\frac{\partial\beta}{\partial y}+\frac{\partial\gamma}{\partial z}=v-2u+u-2v+u+v=0.$$

It is also easily proved that the values of R and S can be expressed by the formulae

$$3R=\left(i\frac{\partial}{\partial x}-k\frac{\partial}{\partial z}\right)r,\quad 3S=\left(j\frac{\partial}{\partial y}-k\frac{\partial}{\partial z}\right)r.$$

3. Let p and q be two independent quaternion solutions of equation (1), then according to the preceding Article we have two equations of the form

$$dp=d\rho\,.\,T+d\sigma\,.\,U,$$
$$dq=d\rho\,.\,V+d\sigma\,.\,W.$$

From these we obtain

$$dp\,.\,U^{-1}=d\rho\,.\,TU^{-1}+d\sigma,$$
$$dq\,.\,W^{-1}=d\rho\,.\,VW^{-1}+d\sigma;$$

whence by subtraction

$$dp\,.\,U^{-1}-dq\,.\,W^{-1}=d\rho\,(TU^{-1}-VW^{-1}),$$

and therefore

$$d\rho=(dp\,.\,U^{-1}-dq\,.\,W^{-1})\,(TU^{-1}-VW^{-1})^{-1}.$$

Similarly we should obtain

$$d\sigma = (dp \,.\, T^{-1} - dq \,.\, V^{-1})\,(UT^{-1} - WV^{-1})^{-1}.$$

Substituting these two values for $d\rho$ and $d\sigma$ in equation (2), we see that it takes the form

$$dr = dp \,.\, P + dq \,.\, Q \qquad\qquad (3),$$

where

$$P = U^{-1} \,.\, (TU^{-1} - VW^{-1})^{-1} \,.\, R + T^{-1} \,.\, (UT^{-1} - WV^{-1})^{-1} \,.\, S,$$

and

$$Q = W^{-1} \,.\, (VW^{-1} - TU^{-1})^{-1} \,.\, R + V^{-1} \,.\, (WV^{-1} - UT^{-1})^{-1} \,.\, S.$$

4. It now only remains to remark that equation (3) is to be regarded as the three-dimensional analogue of the relation

$$dw = f'(z)\, dz,$$

where $w = f(z) = f(x + \iota y)$; which relation expresses that the ratio $dw : dz$ depends only on the origin from which the vector dz is drawn, and not upon its direction. If we have two complex variables $u = x + \iota y$ and $v = z + \iota t$, and if $w = f(u, v)$, then we have a relation of the form

$$dw = \frac{\partial f}{\partial u}\, du + \frac{\partial f}{\partial v}\, dv,$$

where the values of $\partial f/\partial u$ and $\partial f/\partial v$ depend only on x, y, z, t and not upon the values of the ratios $dx : dy$ and $dz : dt$.

In the quaternion theory the order of factors in a product is material, and it turns out that the differential factors must be placed before the finite ones.

The existence of the relation expressed by equation (3), seems to point to the necessity of a discussion of quaternion functions of two variables, *i.e.* involving two variable quaternions. This I hope to be able to furnish in a future communication to the Society.

I am not at present prepared to give a geometrical interpretation of equation (3). Our work furnishes us with materials for calculating P and Q in terms of the differential coefficients of the elements of p, q and r, but the working out of the values by this method would be very tedious. It is possible that when a geometrical interpretation is discovered for equation (3), this may suggest some shorter method of obtaining the required values.

5. In conclusion, we may notice that Laplace's Equation is a particular case of a more general equation to which the quaternion method is applicable, viz. the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial t^2}=0.$$

In this case, as in the former, we have four related solutions, but the equations connecting them are

$$\frac{\partial \delta}{\partial x}=\frac{\partial \gamma}{\partial y}-\frac{\partial \beta}{\partial z}+\frac{\partial \alpha}{\partial t},$$

$$\frac{\partial \delta}{\partial y}=\frac{\partial \alpha}{\partial z}-\frac{\partial \gamma}{\partial x}+\frac{\partial \beta}{\partial t},$$

$$\frac{\partial \delta}{\partial z}=\frac{\partial \beta}{\partial x}-\frac{\partial \alpha}{\partial y}+\frac{\partial \gamma}{\partial t},$$

$$\frac{\partial \alpha}{\partial x}+\frac{\partial \beta}{\partial y}+\frac{\partial \gamma}{\partial z}+\frac{\partial \delta}{\partial t}=0.$$

These four equations are equivalent to the single quaternion equation

$$\left(\frac{\partial}{\partial t}+i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(-\delta+i\alpha+j\beta+k\gamma)=0 \text{ ....(4).}$$

We however require three special solutions of this equation, for which we may take the following:

$$u=t-ix+jy+kz,$$

$$v=t+ix-jy+kz,$$

$$w=t+ix+jy-kz.$$

And, proceeding as in Article 2, we obtain a relation of the form

$$ds=du\,.\,U+dv\,.\,V+dw\,.\,W,$$

where s is any solution of (4), and U, V, W are quaternions whose values depend upon x, y, z, t and not upon dx, dy, dz, dt.* From this we can deduce by proceeding as in the former case that if

* We have

$$2U=\left(\frac{\partial}{\partial t}+i\frac{\partial}{\partial x}\right)s=-\left(j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)s,$$

$$2V=\left(\frac{\partial}{\partial t}+j\frac{\partial}{\partial y}\right)s=-\left(k\frac{\partial}{\partial z}+i\frac{\partial}{\partial x}\right)s,$$

$$2W=\left(\frac{\partial}{\partial t}+k\frac{\partial}{\partial z}\right)s=-\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}\right)s.$$

p, q and r be any three independent solutions of (4), then we obtain a relation of the form

$$ds = dp \,.\, P + dq \,.\, Q + dr \,.\, R.$$

[Since the paper was read I have obtained an analogue for another fundamental theorem concerning functions of a complex variable, viz. the theorem that

$$\int f(z)\, dz = 0,$$

the integral being taken round a closed curve not involving any singular points.

If we integrate over a closed surface, we have

$$\iint (idydz + jdzdx + kdxdy)\, r = \iiint dxdydz \nabla r = 0.$$

This theorem will of course require corrections similar to those necessary to make the other theorem general, but the material for furnishing these is ready to hand. It is to be noticed that in the statement of this theorem, as in that of the theorem of Article 3, the infinitesimal factor has to be placed before the finite one.

There is also a corresponding theorem for the case discussed in Article 5, which may be written in the form

$$\iiint (dxdydz + idydzdt + jdzdxdt + kdxdydt)\, s = 0.]$$

(4) *On a simple model to illustrate certain facts in Astronomy, with a view to Navigation.* By A. SHERIDAN LEA, Sc.D., Gonville and Caius College.

The model consists of a small solid sphere, representing the earth, placed in the centre of a hollow sphere composed of circles of wire. Of these, one represents the celestial equator, one the ecliptic, and the others various meridians corresponding to the parallels of longitude on the earth. Small coloured balls can be attached at any point on the wire circles to represent at any time the positions of the sun or any star relatively to the earth. A wire representing the axis of the ecliptic can be attached to one of the vertical meridians, and this carries the moon with the axis of her orbit inclined at 5° to that of the ecliptic and movable round the latter. The model of the earth is perforated by holes bored at right angles to its surface by means of which a movable horizon, carrying a wire at right angles to it which determines the zenith, can be attached at any point of the earth. The model while

demonstrating the relative movements of the earth, sun, moon and stars is more particularly intended to illustrate and explain the astronomical observations by means of which the position of an observer is determined on the earth's surface, e.g. meridian and ex-meridian altitudes of the sun or a star whether the latter be circumpolar or not. It of course affords a clear explanation of the more important terms used in navigational astronomy, serves also to illustrate the cause of the varying length of day and night at different seasons of the year, the phases of the moon and their relation to the tides, and affords a rough demonstration of the course of solar eclipses.

(5) *Note on a paper relating to the Theory of Functions.* By W. Burnside, M.A., Pembroke College.

In a paper "on the geometrical interpretation of the singular points of equipotential curves" printed in Vol. VI. of the Society's *Proceedings*, Mr Brill has stated some properties of algebraical equipotential curves which are of very doubtful accuracy.

The point of view taken seems to be this: that the several members of an equipotential family of curves do not generally meet in real points at all, and that when they do the points of intersection of the consecutive curves of the family are the branch points of the function which gives rise to the family.

It is not explicitly stated, though it seems to be implied, that these points, viz. the branch points, are the only real points of intersection of such a family. Several of the results obtained in the paper in question depend on the accuracy of the above statements, so that it is perhaps worth while to examine them in some detail.

For this purpose I limit myself to the case in which not only the curves themselves are algebraical, but also the function which gives rise to them. I also consider, except in the last paragraph, real points on the curves and real points only.

Suppose that $f(z, w) = 0$ is an equation of the nth degree in w. If $x + iy$ and $u + iv$ are written for z and w, two equations each connecting x, y, u, v will result from $f = 0$, and by eliminating u and v alternately between these the equations of two systems of algebraical curves will be obtained of the forms

$$f_1(x, y, u) = 0, \quad f_2(x, y, v) = 0.$$

The clearest conception of the intersections (real) of these curves may be obtained in the following manner.

The function w of z, which as defined by the above equation is n-valued, can be represented as a one-valued function on the n-sheeted Riemann's surface belonging to the equation $f = 0$; and

therefore at every point of this surface u and v will each have a single definite value, the only exceptions being the points corresponding to infinite values of w, at which (since w is an algebraical function) u and v will each take all possible values. If then on this surface the curves $u = c\,(-\infty < c < \infty)$ are drawn the separate curves will nowhere meet each other except at the points where w is infinite, and through these points all the curves of the family will pass. The same remark applies to the v-curves. The Riemann's surface on which the curves have been drawn consists in its simplest form of n superposed infinite planes, the continuity between different planes being provided for by cross-cuts (to use Clifford's translation of "Verzweigungsschnitte") connecting properly the branch-points. If now these n planes be regarded as transparent so that the curves may be seen as though lying in one plane, the result will be the same as though they had been so drawn originally. It is then at once clear that through every point of the single x, y plane n u-curves and n v-curves will pass. [It is only by properly taking the curves in pairs so that $u + iv$ is a root of $f(z, w) = 0$ for the value of $x + iy$ considered that the u- and v-curves will cut at right angles.] The only exception will be that through the points $x + iy$, which make one or more values of w infinite, all the curves of both systems will pass. The branch-points, i.e. the points $x + iy$ which make the equation $f = 0$ have equal roots, will be distinguished in this way; that whereas generally the values u_1, $u_2 \ldots u_n$ of the parameters of the n u-curves passing through a particular point are all different; if the point is a branch-point at which r roots of the equation $f = 0$ become equal, r branches of one curve u_1 and $n - r$ other curves $u_{r+1}, \ldots u_n$ will pass through the point; as well, of course, as r branches of a curve v_1 and r other curves $v_{r+1} \ldots v_n$. It is obvious at once that such a family of curves cannot have what is usually called an envelope (real), for this would clearly necessitate the Riemann's surface consisting of an infinite number of sheets.

The point in which Mr Brill seems to have gone astray is the following. In § 3 of his paper he says that if the curves v and $v + dv$ intersect, the necessary condition is

$$f\{u + iv\} = f\{u + i(v + dv)\} \ldots\ldots\ldots\ldots\ldots\ldots (A);$$

[the equation between z and w in the paper referred to is written $z = f(w)$].

This clearly is not *necessary*, for the curves will intersect if

$$f\{u + iv\} = f\{u' + i(v + dv)\} \ldots\ldots\ldots\ldots\ldots\ldots (B),$$

where u and u' are different.

In the light of the above geometrical reasoning the condition (A) is, except at a branch-point, equivalent to supposing that the imaginary part of w has the two different values v and $v+dv$ at one point of the Riemann's surface, which is inconsistent with w being a single-valued function on the surface (unless the point is an infinity of w).

From the condition (A) Mr Brill at once deduces the relation

$$f'(u+iv)=0,$$

on which the statements mentioned at the beginning of this note are based.

The condition (B) implies that the imaginary part of w has the two values v and $v+dv$ at points corresponding to the same $x+iy$ on two different sheets, say 1 and 2, of the Riemann's surface. The values of v on each sheet are continuous and hence on sheet 2 there must be a branch of the curve v indefinitely near the branch $v+dv$. This branch v on sheet 2 and the branch v above considered on sheet 1 will intersect, generally at a finite angle, when the curves are drawn on a single plane; and hence the locus of real intersections of consecutive curves, which it was shewn above could not be an ordinary envelope, is a locus of double points. It is to be noticed that corresponding to the double point considered on the v-curve there will generally *not* be one on a u-curve.

Mr Brill applies his equation

$$f'(u+iv)=0$$

to discuss not only the real but the imaginary intersections of the curves in question. To this purpose the equation appears to me to be entirely inapplicable.

In considering the imaginary intersections of two u-curves, the problem in hand becomes one of functions of *two* complex variables defined by equations like

$$f_1(x, y, u)=0,$$

where x and y may both be complex.

If the attempt be made to deal directly with imaginary values of x and y, say x_1+ix_2, y_1+iy_2 in the original equation defining the function, z becomes $x_1-y_2+i(x_2+y_1)$, and in the analytical work all trace of the original point is lost, as it is impossible to pass back from the two real quantities x_1-y_2 and x_2+y_1 to the original complex quantities which gave rise to them.

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PROCEEDINGS

OF THE

Cambridge Philosophical Society.

January 26, 1891.

PROF. G. H. DARWIN, PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:

(1) *On the Electric Discharge through rarefied gases without electrodes.* By Prof. J. J. THOMSON.

A vacuum tube was exhibited in which an electric discharge was induced by passing the discharge of Leyden jars through a thread of mercury contained in a glass tube coiled four times along it. The induced discharge was found to be confined to the part of the vacuum tube which was close to the primary discharge, and it did not shew striae.

It was also demonstrated that an ordinary striated discharge is strikingly impeded by the presence of a strong field of magnetic force.

(2) *The Laws of the Diffraction at Caustic Surfaces.* By J. LARMOR, M.A., St John's College.

1. One of the most striking phenomena in connection with the propagation of light or other undulations is the circumstance that under certain conditions, common in optics and easily realisable in the case of superficial water waves with varying depth of water, there exists a geometrical boundary beyond which the undulations cannot penetrate at all, but in the neighbourhood of which the disturbance is very much intensified.

In the first approximations of Geometrical Optics, where the undulations are treated as a system of rays, and the energy is considered to be propagated along them, the caustic or envelope of the rays appears as a surface of infinitely great concentration of energy, which is also the boundary of the space into which the energy can penetrate. The conception that must replace this in

a more exact view of the phenomena is that of the theory investigated by Sir George Airy for the case of the rainbow, on the basis of Fresnel's theory of diffraction. It was shown by him that, outside the real caustic of maximum concentration, the energy of the undulations gradually fades away, so that with very minute wave-lengths the boundary of the caustic is quite sharp; but that inside the caustic there is presented a series of successive maxima and minima, in bands running parallel to the absolute maximum or caustic surface.

As the calculations of Sir George Airy had reference chiefly to the phenomena of supernumerary rainbows, he only cared to obtain the relative distances and illuminations of the succession of bands along the asymptote of the caustic.

But the peculiarity of this case of diffraction is that there is no question of an aperture limiting the beam of light, so that the degree of closeness and other relations of the bands must depend only on the character of the caustic surface itself, along which they run. The law, connecting these elements, which is thus suggested for investigation, comes out to be very simple. It appears that for homogeneous light the system of bands is similar to itself all along the caustic, as regards relative positions and relative brightness, and that they are therefore similar to the supernumerary rainbows calculated by Airy and verified experimentally by W. H. Miller; while the absolute breadths at different parts vary inversely as the cube root of the curvature of the caustic surface along the direction of the rays. For different kinds of light the breadths vary as the wave-length raised to the power two-thirds. These laws are exact for the first few bands, usually all that are visible, owing to the extreme closeness of the subsequent ones; they form in fact the physical specification of the nature of caustic surfaces.

2. These statements will be verified in the course of the following analysis of the diffraction near the surface of centres of a wave front, which forms the natural extension of Sir George Airy's investigation for that portion of the caustic which sensibly coincides with its asymptote.

In the first place, taking a cylindrical wave-front, and referring it to the tangent and normal as axes, we have for its equation in the neighbourhood of the origin

$$z = ax^2 + bx^3 + \ldots\ldots$$

The inclination of the tangent at the point x is $\phi = 2ax$, the radius of curvature is

$$R = \left(\frac{d^2z}{dx^2}\right)^{-1} = \frac{1}{2a}\left(1 - \frac{3b}{2a}x\right),$$

and the radius of curvature of the evolute or caustic is

$$\rho = \frac{dR}{d\phi} = -\frac{3b}{(2a)^3}.$$

We have to determine the disturbance, at a point $(\xi, 1/2a)$ in the focal plane of the origin, due to the propagation of this wave. If the amplitude of the motion in the wave-front is $\iota \sin 2\pi t/\tau$ per unit length, the value required for the point in question will be

$$\int \iota \sin\left(\frac{2\pi t}{\tau} - \frac{2\pi r}{\lambda}\right) dS,$$

where for the part in the neighbourhood of the origin

$$r = \left\{(\xi - x)^2 + \left(\frac{1}{2a} - z\right)^2\right\}^{\frac{1}{2}}$$

$$= \gamma^{-1}\{1 - \gamma^2\xi x + \tfrac{1}{2}\gamma^4\xi^2x^2 - \tfrac{1}{2}(\gamma^2 ba^{-1} + \gamma^4\xi + \gamma^6\xi^3)x^3 + \ldots\}$$

correct as far as terms in x^3; where $\gamma^{-2} = (4a)^{-2} + \xi^2$.

This integral is to be taken throughout the extent of the wave-front. The phenomena of optics show however that it is only the parts of the wave-front in the neighbourhood of the normal that are efficient in producing illumination along the normal, for the more remote parts may be blocked out without affecting it. The integral may therefore be confined to the immediate neighbourhood of the origin, and we may proceed by approximation. Taking ξ to be small of the same order as b, we have as far as cubes

$$r = \gamma^{-1} - \gamma\xi x - \tfrac{1}{2}\gamma ba^{-1}x^3,$$

$$ds = dx\,(1 + 2a^2x^2 + 6abx^3),$$

and ι is of the form $\iota = \iota_0(1 + \alpha x + \beta x^2 + \gamma x^3)$,

ι_0 being the amplitude at the origin.

Thus $\displaystyle\int \iota \sin\left(\frac{2\pi t}{\tau} - \frac{2\pi r}{\lambda}\right) dS$

$$= \iota_0 \int \sin\left(\frac{2\pi t}{\tau} - \frac{2\pi}{\lambda\gamma} + \frac{2\pi\gamma\xi}{\lambda}x + \frac{\pi\gamma b}{a\lambda}x^3\right)\{1 + \alpha x + (2a^2 + \beta)x^2 + \ldots\}\,dx.$$

Writing $\quad x' = x + \tfrac{1}{2}\alpha x^2 + \tfrac{1}{3}(2a^2 + \beta)x^3,$

so that $\quad x = x' - \tfrac{1}{2}\alpha x'^2 - \tfrac{1}{3}(2a^2 + \beta)x'^3 + \tfrac{1}{2}\alpha^2 x'^3,$

this becomes

$$\iota_0 \int \sin\left\{\frac{2\pi t}{\tau} - \frac{2\pi}{\lambda\gamma} + \frac{2\pi\gamma\xi}{\lambda}x' - \frac{\pi\gamma\alpha\xi}{\lambda}x'^2 + \frac{1}{\lambda}\left[-\tfrac{2}{3}\pi\gamma\xi(2a^2 + \beta) + \pi\gamma\xi\alpha^2 + \pi\gamma b\right]x'^3\right\} dx',$$

or say

$$\iota_0 \int \sin\left\{\frac{2\pi}{\tau}\left(t - \frac{\tau}{\lambda\gamma}\right) + Ax' + Bx'^2 + Cx'^3\right\} dx',$$

which

$$= \iota_0 D^{-1} \int \sin\left\{\frac{2\pi}{\tau}(t - \text{const.}) + \tfrac{1}{2}\pi(w^3 - mw)\right\} dw,$$

where $w = D\left(x + \dfrac{B}{3C}\right)$, so that $\frac{1}{2}\pi D^3 = C$,

and $$-\tfrac{1}{2}\pi m = \frac{1}{D}\left(A - \frac{B^2}{3C}\right) = (\tfrac{1}{2}\pi)^{\frac{1}{3}} A C^{-\frac{1}{3}}\left(1 - \frac{B^2}{3AC}\right);$$

which gives on reduction, writing unity for $\gamma/2a$,

$$-m = b^{-\frac{1}{3}}\left(\frac{4}{\lambda}\right)^{\frac{2}{3}} 2a\xi\left\{1 - \frac{a}{b}(\tfrac{1}{3}\alpha^2 - \tfrac{5}{18}a^2 - \tfrac{2}{9}\beta)\,\xi\right\},$$

so that

$$2a\xi = m(\tfrac{1}{4}\lambda)^{\frac{2}{3}} b^{\frac{1}{3}}\{1 + \tfrac{1}{2}mb^{-\frac{2}{3}}(\tfrac{1}{4}\lambda)^{\frac{2}{3}}(\tfrac{1}{3}\alpha^2 - \tfrac{5}{18}a^2 - \tfrac{2}{9}\beta)\,\xi\},$$

or in terms of the radius of curvature ($R = 1/2a$) of the wave-front at the origin and the radius of curvature ρ of the caustic

$$\xi = -\rho^{\frac{1}{3}} m\left(\frac{\lambda^2}{96}\right)^{\frac{1}{3}}\left[1 + \tfrac{1}{2}m\left(\frac{3\lambda}{160}\right)^{\frac{2}{3}}\{(\tfrac{1}{12}\alpha^2 - \tfrac{1}{18}\beta)R^2 - \tfrac{5}{18}\}\right].$$

3. Neglecting the term of the second order in this expression

we have $$\xi = -m\left(\frac{\lambda^2\rho}{96}\right)^{\frac{1}{3}},$$

showing that the course of the ray caustic is bordered by a series of fringes which remain similar to each other throughout, and therefore are of the same type as the asymptotic fringes of Airy's supernumerary rainbow; but they come closest together at places of greatest curvature of the caustic according to the law that their separation at any place is proportional to the cube root of the radius of curvature of the caustic at that place.

The investigation shows that unless for fringes at a considerable distance from the ray-caustic their form is not sensibly affected by the varying intensity in the wave-front.

As we proceed along a caustic, the curvature gradually increases and the fringes therefore come together when we approach a cusp. At the cusp itself $b = 0$; and very near to it b is very small, so that to determine the state of matters for an unlimited beam another term would have to be included in the equation of its front; but if the beam is limited in any way the fringes produced by this limitation will there rise in importance, and practically obliterate the ones now under discussion[1].

[1] See Rayleigh, "Investigations in Optics," *Phil. Mag.*, Nov. 1889, pp. 408—10.

4. The results of this analysis indicate how we may proceed in the general case where the wave is not cylindrical, but is curved in two dimensions.

Referred to the normal as axis of z, and the tangents to the arcs of principal curvature as axes of x and y, the equation of its front is

$$z = ax^2 + by^2 + px^3 + 3qx^2y + 3rxy^2 + sy^3 + \dots$$

It is required to find the disturbance propagated to the point

$$\left(\xi,\ 0,\ \frac{1}{2a}\right).$$

Here

$$r_1 = \left\{(\xi - x)^2 + y^2 + \left(\frac{1}{2a} - z\right)^2\right\}^{\frac{1}{2}}$$

$$= \left(\xi^2 + \frac{1}{4a^2} - 2\xi x + x^2 + y^2 - \frac{z}{a}\right)^{\frac{1}{2}}$$

$$= \gamma^{-1} - \gamma\left\{\xi x + \left(1 - \frac{b}{a}\right)y^2 - px^3 - 3qx^2y - 3rxy^2 - sy^3\right\},$$

so that

$$\iint \iota \sin\left(\frac{2\pi t}{\tau} - \frac{2\pi r_1}{\lambda}\right) dx\,dy$$

will be complicated.

But the considerations already mentioned show that the value of the integral is practically settled by the elements in the neighbourhood of the origin, for which x and y are small. We may therefore consider only a small rectangular portion of the wave-front bounded by arcs parallel to the axes of x and y. To determine the diffraction in the plane $z = 1/2a$, we may consider only the plane problem presented by a wave of the form

$$z = ax^2 + px^3;$$

for the uniform curvature in the perpendicular plane represented by the coefficient b will not affect the result at all, as is also obvious on continuing the general calculation. The variation of that curvature represented by the coefficient r will slightly displace the fringes, as it will alter the mean value of $\gamma\xi$.

The dissymmetry indicated by the coefficients q and s will on the average produce no effect on the disturbance at a point in the plane xz; these coefficients introduce odd powers of y which integrated over equal positive and negative range leave no appreciable result.

Thus the illumination in the plane $z = 1/2a$ is determined by the values of a and p only.

Every beam in a homogeneous medium therefore converges to two ray-caustic surfaces which are the two sheets of the surface of centres of curvature of the wave-fronts. Each of these surfaces is physically made up of a series of parallel bright and dark sheets, of which the first is much the brightest, whose distances and relative intensities always retain the same proportions. These distances are at any point proportional to the cube root of the radius of curvature of the normal section of the caustic surface containing the ray which touches it at that point.

5. It is easy enough to obtain actual examples of this general proposition. On looking at a bright lamp, sufficiently distant to be treated as a luminous point, through a plate of glass covered with fine rain-drops, the caustic surfaces after refraction through the drops are produced within the eye itself, and their sections by the plane of the retina appear as bright curves projected into the field of vision. These curves are each accompanied by the other parallel diffraction bands, which separate and become more marked as the curves recede asymptotically, while they assume a different character near the cusps which are a feature of all sections of caustic surfaces. Near these cusps in the cross-section of the caustic surface two different pencils of light come into interference.

These phenomena are quite different from the ordinary cases of entoptic diffraction, in which when the eye is put out of focus by a lens, and a bright sky is viewed through a pinhole, the pencil of light coming through the pinhole projects on the retina shadows of the *muscae volitantes* floating in the aqueous humour, and these are accompanied by the ordinary bands at the boundaries of shadows. This case of an obstacle is the exact complement of that of a similar hole in a screen as regards the position of the bands; so that when the obstacle is small, the diffraction bands round the shadow form exact circles, irrespective of its shape, which is the ordinary visual appearance.

The cusped caustic bands are easily seen when a distant street-lamp is viewed through a spectacle lens with minute rain-drops deposited on it.

The diffraction problem which has here been discussed includes diffraction at a focal line, with an unlimited beam. In practical questions such as those relating to spectroscopes and the Herschelian telescope, the exact focussing would however introduce the nature of the aperture into the discussion, and the limits of the integral would enter.

6. In the case of a cylindrical beam the bands near the caustic have been counted up to 30 or more by W. H. Miller, and the divergence of the more remote ones is too great to allow an

approximate theory like the above to be applied with much certainty. For them, as Prof. Stokes has remarked, a perfectly satisfactory procedure is to simply consider the difference of path of the two pencils of light which reach a given band by different ways; these may be considered as two separate interfering rays, exactly in the manner of Thomas Young's first *aperçu* of the supernumerary rainbow. Now the difference of paths of the two rays up to the point P is clearly the excess of the two tangents from P to the geometrical caustic over the arc between their points of contact. Thus we obtain the following simple and elegant graphical construction, which applies to all the system of bands except the first two or three; imagine the caustic curve constructed as a disc, and let an endless thread be placed round it, the bands will be traced by a pencil strained by this thread in the same manner as in the ordinary construction of an ellipse by a thread passing round its foci; and successive bands will correspond to equal increments in the length of the thread.

(3) *The effect of Temperature on the Conductivity of Solutions of Sulphuric Acid.* (Plates IV. and V.) By Miss H. G. KLAASSEN, Newnham College (communicated by Prof. J. J. THOMSON).

Graham has shown that the viscosity of sulphuric acid increases upon addition of water until a maximum is reached at the composition of nearly one molecule of water to one of acid. Upon further dilution the viscosity continually decreases.

If a curve, in which the ordinates represent the electrical resistance and the abscissae the percentage composition of solutions of sulphuric acid in water, be drawn from Kohlrausch's observations, it will be seen that a point of maximum resistance exactly corresponds to this degree of concentration. The fact, that these maxima should both occur when the composition of the solution is almost exactly one molecule of water to one of acid, points to the existence of the hydrate $H_2SO_4 . H_2O$.

If the increase of viscosity and of resistance, supposing it due to this hydrate, were found to diminish with a rise of temperature, this fact would furnish strong evidence in favour of the theory that the hydrate $H_2SO_4 . H_2O$ dissociates into H_2SO_4 and H_2O at higher temperatures.

That the increased viscosity does diminish with the rise of temperature has been proved by some recent determinations of the viscosity of sulphuric acid solutions at temperatures between 15° C. and 100° C.[1]

[1] *Phil. Mag.*, Oct. 1889.

It was suggested to me by Professor Thomson that it would be of interest to ascertain if the increased electrical resistance due to the hydrate would also tend to disappear at higher temperatures.

With this object in view I determined the resistance of various solutions of sulphuric acid at temperatures between 15° C. and 100° C.

The resistance was measured by means of the Wheatstone bridge with a Post Office resistance box and a double commutator[1] which reversed the battery and galvanometer circuits simultaneously. In every experiment the solution was heated twice to a temperature of nearly 100° C., but in no case did the observations taken indicate any change of concentration from absorption of steam.

The following observations were taken:

97 % H_2SO_4

Temperature	Resistance	Temperature	Resistance
17·4° C.	27·4 Ohms	86·5° C.	7·7 Ohms
97·6	6·8	85·5	7·8
86·6	7·7	80·4	8·3
79·2	8·4	71·2	9·4
70·9	9·5	59·1	11·2
59·0	11·1	46·5	14·0
48·2	13·5	42·7	15·0
37·9	16·5	35·6	17·5
29·5	20·0	30·2	19·8
17·0	27·9		

95 % H_2SO_4

Temperature	Resistance	Temperature	Resistance
16·2° C.	20·3 Ohms	64·0° C.	7·6 Ohms
90·9	5·37	48·2	9·8
87·0	5·6	34·1	13·0
78·7	6·22	20·6	18·1
70·5	6·9	18·9	18·85
61·0	7·9	90·0	5·4
49·5	9·6	94·5	5·1
39·6	11·7	81·1	6·1
		67·0	7·3

[1] For description of commutator, see *Brit. Ass. Report*, 1886, p. 328.

90·4 °/。 H_2SO_4

Temperature	Resistance	Temperature	Resistance
16·4° C.	19·22 Ohms	58·0° C.	7·1 Ohms
81·4	4·9	49·2	8·4
79·9	5·0	47·7	8·7
67·3	6·1	37·1	11·0
92·25	4·2	29·1	13·4
90·6	4·3	17·1	18·82
76·9	5·22	89·2	4·4
66·9	6·1	75·6	5·3
50·5	8·3	50·7	8·2
37·5	10·9	28·9	13·5
39·0	10·5	14·2	20·5
67·4	6·06	24·7	15·0

86·5 °/。 H_2SO_4

Temperature	Resistance	Temperature	Resistance
14·7° C.	22·2 Ohms	94·2° C.	4·0 Ohms
34·6	12·18	92·4	4·1
66·9	6·1	85·2	4·5
64·8	6·3	72·5	5·5
44·0	9·6	61·9	6·6
35·1	12·0	47·9	8·8
16·2	21·1	23·45	16·7

84·1 °/。 H_2SO_4

Temperature	Resistance	Temperature	Resistance
15·7° C.	21·62 Ohms	68·5° C.	5·8 Ohms
24·65	16·51	86·9	4·4
35·05	11·9	16·6	21·1
43·4	9·7	38·7	10·95
54·2	7·7	66·4	6·05
62·8	6·4	54·8	7·6

81·7 °/。 H_2SO_4

Temperature	Resistance	Temperature	Resistance
17·3° C.	19·85 Ohms	16·3° C.	20·3 Ohms
67·7	5·7	57·2	6·9
59·5	6·6	39·1	10·4
47·0	8·6	37·5	10·8
29·9	13·3	15·9	20·6
87·2	4·1	87·2	4·1
92·0	3·83	80·5	4·6
81·9	4·5		
80·5	4·6		

74·7 °/。 H_2SO_4

Temperature	Resistance	Temperature	Resistance
16·8° C.	13·3 Ohms	91·9° C.	3·35 Ohms
96·7	3·2	87·9	3·5
93·6	3·3	81·8	3·8
82·5	3·8	71·4	4·4
80·5	3·9	50·2	6·2
68·9	4·6	42·0	7·3
50·3	6·2	30·5	9·4
42·0	7·3	28·0	10·0
30·5	9·4	60·4	5·2
28·7	9·8	63·2	4·97

30 °/。 H_2SO_4

Temperature	Resistance	Temperature	Resistance
17·0° C.	2·68 Ohms	63·6° C.	1·53 Ohms
42·0	1·89	46·0	1·81
62·4	1·60	15·8	2·7
91·3	1·33	·6	3·67
84·0	1·40		

These observations are represented graphically in Plate IV. in which Temperatures are represented by ordinates, and Resistances by abscissæ.

The form of these curves is similar in type to that of the corresponding ones for viscosity given by Mr D'Arcy in the *Phil. Mag.*, Oct. 1889. The curvature first increases with the temperature, reaches a maximum, and then decreases.

In Plate V. are represented the isothermals for 18° C., 20° C., 30° C., 40° C., 50° C., 60° C., 70° C., 80° C. and 90° C., drawn from the curves in Plate III., resistances being represented by ordinates, and the percentages of H_2SO_4 by abscissæ. Points marked ⊕ are taken from Kohlrausch's observations.

The isothermal for 18° C. has been continued up to its minimum point (30 °/。 H_2SO_4) from the observations of F. Kohlrausch.

This curve has a point of minimum resistance at about

92 °/。 H_2SO_4,

the resistance then rises, reaching a maximum at

84·5 °/。 H_2SO_4 (the hydrate $H_2SO_4 . H_2O$),

and then again falls. The isothermals for higher temperatures show that this rise in the resistance gradually diminishes, up to 70° C., above which temperature the resistances are in descending order of magnitude. The increased resistance due to the hydrate has, however, not quite disappeared, for even in the isothermal for 90° C. there is a change in the sign of the curvature, although there is no longer a point of maximum resistance.

Additional evidence in favour of the existence of the hydrate $H_2SO_4 . H_2O$ is afforded by the fact that it can be obtained in the crystalline form.

The hydrates of SO_3 which have at present been crystallised are:

			melting point	
(1)	$H_2S_2O_7$	......		
(2)	H_2SO_4	......	0° C.	
(3)	$H_2SO_4 . H_2O$	......	7·5° C.	
(4)	$H_2SO_4 . 4H_2O$	......	−25° C.	(Pickering, *Chem. News*, 1889).

We know from the experiments of W. Kohlrausch upon solutions of SO_3 in H_2SO_4, and from those of F. Kohlrausch upon aqueous solutions of H_2SO_4, that the formation[1] of (2) and (3) is accompanied by an increased electrical resistance. $H_2SO_4 . 4H_2O$, on the other hand, appears to have no effect upon the resistance at ordinary temperatures. What effect the formation of the hydrate $H_2S_2O_7$ may have upon the resistance has not been ascertained, as the observations of W. Kohlrausch do not extend to solutions of greater concentration than 90·67 °/$_\circ$ SO_3.

Judging from these facts it would seem that the formation of a hydrate does not necessarily produce an increased resistance, though in the cases of H_2SO_4 and $H_2SO_4 . H_2O$ it appears to do so.

But we have seen that the hydrate $H_2SO_4 . H_2O$ dissociates as the temperature rises. It is possible, therefore, that $H_2SO_4 . 4H_2O$ which has a melting point of − 25° C., while that of $H_2SO_4 . H_2O$ is 7·5° C., may be so far dissociated at 18° C. that there is not sufficient of the hydrate present to produce a perceptible effect upon the resistance.

February 9, 1891.

PROFESSOR G. H. DARWIN, PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:

(1) *On Rectipetality and on a modification of the Klinostat.* By DOROTHEA F. M. PERTZ and FRANCIS DARWIN, M.A., Christ's College.

[*Abstract; received March* 5, 1891.]

Vöchting has shown that geotropically induced curvatures may be removed by subjecting the curved plant to slow rotation on a horizontal klinostat. It is usually assumed that the growing part, being freed from external stimulation, straightens itself by

[1] The point of maximum resistance is between 99·75 °/$_\circ$ and 99·9 °/$_\circ$ H_2SO_4.

an inherent regulating power to which Vöchting has given the name *Rectipetality*. But it is not certain that the klinostat does remove external stimuli. Elfving's experiments on the growth of grass-halms show that though the gravitation-stimulus is symmetrically distributed, it is not destroyed. If the klinostat's action depends on the symmetrical distribution of stimuli, not on their removal, we shall be compelled to take a different view of rectipetality. To test this question a new form of horizontal klinostat was devised. The axis of this instrument is not kept in constant rotation, but at intervals of half-an hour it executes a half revolution. A shoot or stem geotropically curved is fixed in the klinostat so that the plane of curvature is horizontal. The succession of half turns prevents any geotropic distortion in the plane at right angles to the original plane of curvature, while in that plane the plant is free to increase or diminish its curvature apart from any fresh gravitation-stimulus. The experiments show that under these circumstances the curvature diminishes, and this can only be due to an inherent regulating power, the rectipetality of Vöchting.

The modified klinostat, which will probably be of use for other purposes, was designed and made by the Cambridge Scientific Instrument Company.

(2) *On the Occurrence of* Bipalium Kewense, *Moseley, in a new Locality; with a Note upon the Urticating Organs.* By ARTHUR E. SHIPLEY, M.A., Christ's College.

[*Received February* 9, 1891.]

Bipalium Kewense was first described by Professor Moseley in the year 1878, from specimens obtained in the hot-houses at Kew. In 1883 Dr Günther received some examples of this species from Welbeck Abbey, and one specimen was described from Clapham Park. Mr O. Salvin exhibited some of these animals before the Zoological Society in 1886, collected from amongst pieces of broken tiles at the bottom of some pots of Calceolarias which had stood in a cold frame all the winter in his garden near Haslemere, Surrey. Some specimens from the same locality formed the object of some interesting observations by Professor Jeffrey Bell during the same year[1]. Finally the specimens which I am able, owing to the kindness of Professor Newton, to exhibit to the Society this evening came from the neighbourhood of Bath.

[1] Since writing the above, Prof. Herdman has informed me that *Bipalium Kewense* was found in an Orchid house at Aigburth, near Liverpool in 1888, and Mr Beddard writes to me that specimens are from time to time brought to him from the gardening department of the Zoological Gardens.

Outside England this species has been found in the Orchid-Houses of the Botanical Garden in Berlin, and in the Palm-Gardens of Frankfort. As early as 1883 it had been noticed in the Botanic Garden of Cape Town, and it has been known in Sydney since 1874, although its presence was not registered till 1888. It has recently been found by Mr J. J. Lister under stones in the forests of Upolu, Samoa, and it is not impossible that this is one of its native habitats[1].

Unfortunately the records of the appearance of *B. Kewense* fail to throw much light on its native habitat. It has almost invariably turned up in hot-houses, usually associated with Orchids; and there is nothing to show that these Orchids in all cases came from the same part of the globe. Sydney is the only place where it seems to have established itself, and here it is described as existing in great numbers, lying under pieces of wood, etc., or crawling along the pavements and palings. In England, as is pointed out above, it has only occurred sporadically and a few at a time, probably introduced afresh, or disseminated from Kew. It does not seem in any danger of establishing itself. *Rhynchodemus terrestris*, O. F. Müll, still remains the only British Land Planarian.

The specimens from Bath vary a good deal in length. One, whilst crawling up a wall, attained the length of seven inches. The body is extremely extensile and soft, and the animals seem capable of creeping through the smallest crannies. Those that Professor Moseley described were some of them nine inches long, a length surpassing that of any other species of *Bipalium*, but New South Wales specimens were even longer; one of them measured 14 in. and another 9 in. (6). The worms crawl about actively, by means of the strong cilia on their ventral surface, and in their natural conditions appear to coil round stems and blades of grass, etc.

The semilunar anterior end, which is characteristic of the genus, does not always maintain its outline, but the contour of the head is constantly changing (1). The head is generally raised above the surface of the ground, and processes appear to be pushed out from its edge, which test the surface upon which the animal crawls.

There seems to be no doubt that *B. Kewense*, like other allied species, is nocturnal in its habits, strong sunlight being harmful and often fatal to it. Its surroundings must also be kept very damp. One of the most curious features of its economy is the harmful effect which certain foreign substances have upon its well-being. Dr Trimen (27) records how one specimen was killed in

[1] Vide *Zoolog. Anzeiger*, No. 361, p. 139.

the space of some minutes by dropping upon a grate covered with blacking; Mr Fletcher (6) states that in Sydney, where they appeared in considerable numbers, they were constantly found either moribund or dead on the pavement, the surface of which apparently did not agree with them, and the living specimens which were kindly sent to Cambridge by Mr L. Birch unfortunately crawled on to the sides of a glass bottle and there died. Mr Harmer tells me that the same fate overtook some specimens of *Rhynchodemus terrestris* which had found their way on to a glass surface. This extreme sensibility to the contact of foreign substances seems strange when we remember what a copious coating of mucus these animals can produce at will.

Bipalium Kewense lives on earthworms, insects, etc. and probably, like the other species of the same genus whose habits have been investigated, it is entirely carnivorous, and does little harm to the orchids and other plants with which it may be associated. Some species devour small molluscs as well as earthworms, the radula of a snail having been found within a *Bipalium* from the Philippines (12). *B. javanum* (14) eats small Gastropods; its pharynx, which envelopes the body of the animal, shell and all, being powerful enough to crush the shell, the pieces of which do not pass into the alimentary canal, and are either rejected or possibly dissolved by the secretion of the numerous glands which open into the pharynx. At present there is no evidence to show that these animals are ever vegetable feeders.

Like most other Turbellarians, *Bipalium* is hermaphrodite; but it also reproduces by transverse fission, and this may to some extent account for the great variation in length in individuals found in the same locality, and also for the not unfrequent absence of the semilunar head. *B. javanum* is protandrous, the male organs being mature in July and August, the egg-cocoons not being deposited till October or November (14).

The faculty of depositing considerable quantities of mucus is one of the most remarkable characteristics of these animals; their path can be traced by a slimy tract which quickly dries up. Prof. Jeffrey Bell's (1) observations tend to show that in *Bipalium Kewense* the mucus is secreted by the anterior end of the body: this observer is of opinion that the secretion may serve to entangle offending bodies, and possibly also helps to catch objects which serve as food. In some of the tropical species this mucus is very copious, and hardens into threads by means of which the animals suspend themselves. They are occasionally blown, hanging at the end of their threads, from one stem or branch to another, like the young of many species of spiders.

The specimens which were sent to Cambridge had unfortunately crawled on to the inner surface of the glass bottle in

which they were confined. When they came into my hands they were in a state of deliquescence, and formed a slimy mass with a more or less definite outline. On examining this deposit with a microscope, it became apparent that the various cells composing the body of the animal had parted company. Amongst the numerous glandular and other cells which formed a considerable portion of the slime, certain large cells were seen, each of which appeared to contain two or more bodies. At first I mistook these structures for the ordinary rod-shaped rhabdites, so common in Planarians. On a more thorough examination, however, I found that what I had at first taken for two bodies was in fact but one continuous structure bent in the form of a V; and that to one end of the body a long whip-like appendage was fixed. Several of the cells, which were provided with a nucleus, burst while under observation; and they appeared to do so by the efforts of the contained rods to straighten themselves; at any rate, when the cell burst, the rods straightened themselves, and the thread borne by their ends was thrown out. The thread did not, like the threads of the nematocysts of *Hydra*, stretch straight out, but assumed a somewhat coiled disposition. The rhabdites, which were also present, had very much the appearance of the thicker basal portion of the flagellated structures, but were without the whip; and it occurred to me that possibly the rhabdites, which are so common in the *Turbellaria*, might be derived from the basal part of the flagellated structures. This view is, to a slight extent, supported by the fact observed by Loman that, in the East Indian species he examined, the number of the simple rhabdites and the number of flagellated structures varied inversely. Loman is inclined to regard the somewhat similar bodies, prolonged at each end into a fine thread, which he found in his *Bipalium javanum*, as true nematocysts, such as are found in Coelenterates and in some species of Rhabdocoels, e.g. in *Microstoma lineare* and *Stenostoma Sieboldii*. But whereas in the typical nematocyst the urticating thread is coiled up inside the capsule, and is evaginated when shot out, in the bodies found in *Bipalium* there is no capsule, but a basal thick portion, either bent or coiled, and a thin thread wound round this. That these organs have, at any rate in some species, the same irritating properties as the nematocysts of Coelenterates is shown by the fact that Mr Thwaites (18) experienced, when he applied his tongue to some living land Planarians in Ceylon, "a feeling of unpleasant tingling," which "was accompanied with slight swelling. The sensations [were] very similar to what is experienced upon a slight scalding." Mr Dendy also tasted the Australian species, *Geoplana Spenceri*, and describes the results as very unpleasant.

The following is a list of the more important papers which

have appeared on Land Planarians since the publication of Professor Moseley's memoir (17).

1. Bell, F. Jeffrey;
Note on Bipalium Kewense, and the Generic Characters of Land-Planarians.
Proc. Zool. Soc., 1886, p. 166.

2. Bergendal, D.;
Zur Kenntniss der Landplanarien.
Zool. Anzeiger, x. *Jahrgang*, 1887, p. 218.
— Do. *Ann. and Mag. Nat. Hist.* (5th Ser.), Vol. 20, p. 44.

3. Blomefield, L.;
Note on the Occurrence of the Land Planaria (Planaria terrestris) in the neighbourhood of Bath.
Proc. Bath Nat. Hist. and Antiquarian Field Club, Vol. iii., 1887, p. 72.

4. Carrière, J.;
Ein neuer Fundort von Planaria terrestris, O. F. Müller.
Zool. Anzeiger, ii. *Jahrgang*, 1879, p. 668.

5. Dendy, Arthur;
Anatomy of an Australian Land Planarian.
Trans. Roy. Soc. Vict., Vol. i. Pt. ii., p. 50.

6. Fletcher, J. J.;
Remarks on an introduced species of Land Planarian, apparently Bipalium Kewense, Moseley.
Proc. of the Linnean Soc. of New South Wales, 2nd Ser., Vol. ii., 1887, p. 244.

7. Fletcher, J. J., and Hamilton, A. G.;
Notes on Australian Land Planarians with Descriptions of some New Species.
Proc. of the Linnean Soc. of New South Wales, 2nd Ser., Vol. ii., p. 349.

8. v. Graff, L.;
Ueber ein. interessante Thiere des Zoolog. und Palmengartens zu Frankfurt a. M.
Der Zoolog. Garten, 20 *Jahrgang*, 1879, p. 196.

9. Harmer, S. F.;
[Note on the occurrence of Rhynchodemus terrestris at Cambridge.]
Proc. of the Camb. Philos. Soc., Vol. vii., Pt. ii., p. 83.

10. v. Kennel, J.;
Die in Deutschland gefundenen Land-Planarien, Rhyncodemus terrestris, O. F. Müller, und Geodesmus bilineatus, Mecznikoff.
Arbeiten aus dem Zoologisch-Zootomischen Institut in Würzburg, Bd. v., 1882, p. 120.

11. — Bemerkungen über einheimische Land-Planarien.
Zool. Anzeiger, i. *Jahrgang*, 1878, p. 26.

12. Loman, J. C. C.;
Ueber den Bau von Bipalium, Stimpson.
Bijdragen tot de Dierkunde, *Aflev.* 14, p. 61, 1887.

13. — Zwei neue Arten von Bipalium.
Zool. Anzeiger, Vol. vi., 1883, p. 168.

14. — Over den bouw van de Land-Planariën.
Tijdschr. Nederl. Dierk. Ver., i. Deel, p. 130.

15. — Ueber neue Landplanarien von den Sunda-Inseln.
Zool. Ergebn. ein. Reise in Niederl. Ost. Ind. Heft. i. p. 131.

16. Moseley, H. N.;
Description of a new Species of Land-Planarian from the Hot-houses at Kew Gardens.
Ann. and Mag. of Nat. Hist., 5th Ser., Vol. i., 1878, p. 237.

17. MOSELEY, H. N.;
On the Anatomy and Histology of the Land Planarians of Ceylon, with some Account of their Habits and a Description of two new Species, and with Notes on the Anatomy of some European Aquatic Species.
Phil. Trans., 1873, p. 105.

18. —;
Urticating Organs of Planarian Worms.
Nature, Vol. XVI., 1877, p. 475.

19. — Notes on the Structure of several Forms of Land Planarians, etc.
Quart. Journ. Microsc. Sci., Vol. XVII., 1877, p. 273.

20. DE MAN, J. G.;
Geocentrophora sphyrocephala n. gen. n. sp., eene landbewonende Rhabdocœle.
Tijdschr. Nederl. Dierk. Ver., II. Deel, 1875.

21. — De gewone europeesche Land-planarie, Geodesmus terrestris O. F. Müll.
Tijdschr. Nederl. Dierk. Ver., II. Deel, 1876, p. 238.

22. RICHTERS, FERD.;
Bipalium Kewense eine Landplanarie des Palmenhauses zu Frankfurt a. M.
Der Zoolog. Garten, XXVIII. Jahrgang, 1887, p. 23.

23. SALVIN, O.;
Exhibition of Bipalium Kewense.
Proc. Zool. Soc., 1886, p. 205.

24. SCHULZE, F. E.;
Über lebende Bipalium.
Sitzungsber. Ges. Nat. Freunde Berlin, 1886, p. 159.

25. SPENCER, W. B.;
Notes on some Victorian Land Planarians.
Proc. Royal Soc. of Vict., 1891, p. 84.

26. STEENSTRUP, J.;
Om Jord-Fladormens (Planaria terrestris O. F. M.) Forekomst i Danmark.
Vid. Medd. f. d. naturh. Forening i Kjöbenhavn, 1869, p. 189.

27. TRIMEN, ROLAND;
On Bipalium Kewense at the Cape.
Proc. Zool. Soc., 1887, p. 548.

28. VEJDOVSKÝ, F.;
Note sur une nouvelle Planaire terrestre (Microplana humicola nov. gen., nov. sp.).
Rev. Biol. Nord de la France, II., 1889–90, p. 129.

29. ZACHARIAS, O.;
Landplanarien auf Pilzen.
Biolog. Centralblatt, 8 Bd., 1888–1889, p. 542.

(3) *The Medusæ of* Millepora *and their relations to the medusiform gonophores of the* Hydromedusæ. By S. J. HICKSON, M.A., Downing College.

[*Abstract; reprinted from the Cambridge University Reporter, Feb.* 17, 1891.]

In *Millepora plicata* no medusiform structures of any kind were observed. The spermaria are simple sporosacs on the sides of the dactylozooids. The eggs are extremely minute and show frequently amœboid processes: they are found irregularly dis-

tributed in the cœnosarcal canals of the growing edges of the colony.

In *Millepora murrayi* from Torres Straits large well-marked medusæ, bearing the spermaria, were observed lying in ampullæ of the cœnosteum. Even when free from the cœnosarcal canals and ready to escape they show no tentacles, sensory bodies, radial or circular canals, velum or mouth. They are formed by a simple metamorphosis of a zooid of the colony. The eggs of this species, like those of *M. plicata*, are extremely small and amœboid in shape. They are not borne by special gonophores.

In the *Stylasteridæ*, the eggs are large, contain a large quantity of yolk, and are borne by definite cup-like structures produced by foldings of the cœnosarcal canals.

In *Allopora* the spermarium is enclosed by a simple two layered sac composed of ectoderm and endoderm. The endoderm at the base is produced into the centre of the spermarium as a simple spadix.

In *Distichopora* the male gonophores are similar to those of *Allopora*, but there is no centrally placed endodermal spadix. In both genera a two layered tube (seminal duct) is produced at the periphery of the gonophore when the spermatoza are ripe.

Neither the gonophores of *Allopora* and *Distichopora*, nor the medusæ of *Millepora murrayi* show any traces in development of being degenerate structures like the adelocodonic gonophores of the other *Hydromedusæ*.

(4) *The Development of the Oviduct in the Frog.* By E. W. MacBride, St John's College.

[*Abstract: received February* 9, 1891.]

In July of last year I undertook, at Mr Sedgwick's suggestion, the investigation of the origin and growth of the oviduct in the Frog. Some considerable time was spent in determining the stages in which it is formed. No trace of it is visible till the animal has lost all the characteristic tadpole organs except the tail; and it is complete, from a morphological point of view, when the frog has attained a length of about 17 millimetres, after the tail has been absorbed.

So far as I am aware, there are only two papers dealing with the development of this duct in the Anura. The first of these is by C. K. Hoffmann, in the 'Zeitschrift für wissenschaftliche Zoologie' for 1886 (Bd. 44), and the other by A. M. Marshall and E. J. Bles, in the second volume of the 'Studies from the Biological Laboratories of the Owens College.'

Hoffmann describes the duct as arising from a patch of modified peritoneum just ventral to the third and now only remaining

nephrostome of the pronephros. This patch, he says, is dorsally involuted to form a groove, open below. Ventrally it is prolonged downwards and outwards over the surface of the pronephros, and even beyond it for a short distance. It is distinguished from the unmodified epithelium by being highly columnar. The part of the Wolffian duct in front of the mesonephros, the lumen of which is at this stage much reduced, then separates itself from the degenerating pronephros, and splits into two rods of cells. The dorsal of these is continuous with the Wolffian duct behind, and the ventral one applies itself in front to this involuted patch of peritoneum, and forms the first rudiment of the oviduct. But the oviduct does not grow back in continuity with the Wolffian duct. On the contrary, it enters into close connection with a longitudinal strip of peritoneum which lies to the outer border of the kidney, and consists of columnar epithelium. Hoffmann states that the hinder portion of the duct is formed from these cells, though whether by involution to form a groove or by proliferation he could not determine. In the meantime, the groove which formed the ostium of the duct, and which was originally dorsal, has become prolonged ventrally round the base of the lung. It closes, forming a canal which now opens ventrally. Later, this ventral extension atrophies, leaving the ostium in its original dorsal position.

Marshall and Bles have only observed a few isolated facts in the development of the oviduct. They confirm Hoffmann in his account of the ventral displacement of the ostium; but fail to observe any splitting of the front part of the Wolffian duct. They state that the hind end of the duct is, in the first year, a solid rod of cells; but do not notice any relation to the peritoneum.

My observations differ from those of Hoffmann in several important particulars. It will be convenient to describe first the origin and fate of the abdominal opening, and then that of the rest of the duct. In a tadpole in which the hind-limbs alone are visible, I find three nephrostomes in the head-kidney, the cells of which bear long flagella pointing inwards, as Hoffmann has pointed out. The first of these is situated some way in front of the glomerulus, the second immediately in front of the attachment of the glomerulus, and the third immediately behind it.

In my next stage; that is to say in a frog 38 mm. long (including the tail, which was about 21 mm. long), one nephrostome only remains; but, as this is situated some considerable distance in front of the glomerulus, it must be regarded as the first and not the third of the preceding stage. Immediately ventral to it there is a groove in the peritoneum, open below, the epithelial lining of which is very columnar and quite different in appearance to that of the nephrostome. This columnar epithelium is con-

tinued out over the surface of the pronephros and beyond it, as Hoffmann has described. The groove, traced backwards for 15 sections, becomes a canal, which after two sections ends in a solid thickening of the peritoneum. In succeeding stages the groove extends ventrally, as described by Hoffmann, and in the last stage it has become a canal, opening somewhat ventrally, the opening having already acquired the fimbriæ of the adult orifice. But I have been quite unable, after examining a number of adults, to detect any important difference between the position of the abdominal opening in my last stage and that of the adult funnel. The latter is not situated dorsally but at the side of the lung. It is fimbriated, and these fimbriæ are continued, lying in a groove, on to the mesentery connecting the stomach and liver, over the ventral surface of the lung. It is true that the length of the orifice appears to have extended somewhat in a dorsal direction, but that is all.

As to the origin and backward growth of the duct behind the funnel, I find that the mode of origin described by Hoffmann for its hinder portion holds for its entire length. It has been mentioned above that in the earliest stage the peritoneal groove, traced backwards, ends in a slight thickening of the peritoneum. The next stage shows the same condition of things; but a good length of the groove has been converted into a canal. In the first stage the Wolffian duct is still discernible in the part of the body in front of the mesonophros, but it is reduced to a rod of pale degenerate cells. There is no indication of any splitting such as Hoffmann described; and it appears *à priori* in the highest degree improbable that fresh development should take place from such an atrophied rudiment. I may mention that Hoffmann omits to give any figures illustrating this point. In frogs of this age (that is to say those in which the tail is about as long as the body), there is a distinct line of epithelium on the outer border of the kidney, reaching back to the cloaca, which is distinguished from the rest of the peritoneum by its more columnar character. In succeeding stages, all trace of the Wolffian duct in front of the kidney is gone; and the thickening of the peritoneum mentioned above travels back along this line of modified epithelium. I have called it a peritoneal thickening because I believe it to be derived from the peritoneum. It appears in sections as a projecting nodule of deeply staining tissue, the outermost cells of which pass at the side into the ordinary peritoneum. Furthermore, although the rudiment in front of the kidney grows back with some regularity, behind the kidney it is formed long before one can trace it at the side of this organ. It nowhere comes in contact with the Wolffian duct. But it is not possible to speak with absolute certainty as to the origin of the cells which compose this solid rudi-

ment, because there is, especially in front, some lymphoid tissue at the outer border of the kidney—in fact the whole pointed end of the mesonephros degenerates to a string of such tissue.

The lumen of the duct appears first in front and then behind in the region of the kidney. In the latter position it appears here and there in patches. It is formed by the rearrangement of some of the cells in the rod in a stellate manner, sometimes one cell and sometimes two cells deep beneath the surface.

The conclusion which seems to be suggested by these investigations is the complete independence of the oviduct from the Wolffian duct in the Anura.

I have, in conclusion, to express my warmest thanks to Mr Sedgwick for his advice and assistance to me in this work.

February 23, 1891.

PROF. G. H. DARWIN, PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:

(1) *Tidal Prediction—a general account of the theory and methods in use and the accuracy attained.* By Prof. G. H. DARWIN.

Published in *Nature*, Vol. 43, p. 609.

(2) *On Quaternion Functions, with especial Reference to the Discussion of Laplace's Equation.* By J. BRILL, M.A., St John's College.

1. The following communication is intended as a sequel to the one that I made to the Society at the end of last term. In that paper I showed how we might obtain analogues to the theorems connected with conjugate functions with the aid of four related solutions of Laplace's Equation obtainable from the solution of a quaternion differential equation of the first order. I now propose to obtain a form for the general integral of the said equation.

2. On account of the non-commutative character of the symbols involved, quaternion functions are of a more complicated character than ordinary scalar functions, and for their full discussion would require a notation and nomenclature of their own. We may, however, in the case of functions of a single quaternion, as was done by Hamilton in the case of the exponential, extend the definitions of some of the ordinary scalar functions so as to apply to quaternions, by defining the quaternion function as the sum of a quaternion series exactly similar in form to the scalar series which defines the corresponding scalar function. It is to be remarked, however, that this method cannot be consistently carried

out to the end, as the inverse of a quaternion function would not in general correspond with the quaternion function framed on the model of the inverse scalar function. Still further difficulties would arise if we attempted to apply the scalar notation to functions of two quaternions.

So far as I am aware, Boole[1] was the first to give a general expression for a function of a single quaternion framed on the model of a specified scalar function. His expression may be easily deduced from Sylvester's Interpolation Formula[2] in the Theory of Matrices, which states that if $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$ be the latent roots of an n-ary matrix, then

$$f(m) = \Sigma \frac{(m-\lambda_2)(m-\lambda_3)\ldots\ldots(m-\lambda_n)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots\ldots(\lambda_1-\lambda_n)} f(\lambda_1),$$

where m denotes the matrix, provided that none of the latent roots are equal.

The identical equation satisfied by a quaternion is

$$q^2 - 2qSq + (Tq)^2 = 0,$$

and, therefore, the latent roots of the quaternion are given by the equation

$$\lambda^2 - 2\lambda Sq + (Tq)^2 = 0;$$

and since
$$(Tq)^2 = (Sq)^2 + (TVq)^2,$$

it is clear that the roots are

$$Sq + \iota TVq \text{ and } Sq - \iota TVq.$$

These roots are obviously distinct except in the case when q reduces to a scalar. Thus we have

$$\begin{aligned} f(q) &= \frac{q-\lambda_2}{\lambda_1-\lambda_2} f(\lambda_1) + \frac{q-\lambda_1}{\lambda_2-\lambda_1} f(\lambda_2) \\ &= \frac{Vq+\iota TVq}{2\iota TVq} f(Sq+\iota TVq) - \frac{Vq-\iota TVq}{2\iota TVq} f(Sq-\iota TVq) \\ &= \frac{1}{2}\{f(Sq+\iota TVq) + f(Sq-\iota TVq)\} \\ &\qquad + \frac{1}{2\iota} UVq\,\{f(Sq+\iota TVq) - f(Sq-\iota TVq)\}, \end{aligned}$$

which is Boole's result.

1 "On the Solution of the Equation of Continuity of an Incompressible Fluid," *Proc. R. I. A.*, VI. 375—385.

2 This theorem was stated by Sylvester in a paper in the *Phil. Mag.* for October 1883, entitled "On the Equation to the Secular Inequalities in the Planetary Theory." It also appears as the second law in his "Laws of Motion in the World of Universal Algebra."

3. Assuming Boole's result and writing $f'(q)$ for the quaternion function framed on the model of $f'(x)$, we easily obtain

$$\begin{aligned}\nabla f(q) &= Sf'(q)\,.\,\nabla Sq - TVf'(q)\,.\,\nabla TVq + TVf(q)\,.\,\nabla UVq \\ &\qquad + \{TVf'(q)\,.\,\nabla Sq + Sf'(q)\,.\,\nabla TVq\}\,.\,UVq \\ &= \{\nabla Sq + \nabla TVq\,.\,UVq\}\,\{Sf'(q) + UVq\,.\,TVf'(q)\} \\ &\qquad + TVf(q)\,.\,\nabla UVq \\ &= \nabla q\,.\,f'(q) + \nabla UVq\,.\,\{TVf(q) - TVq\,.\,f'(q)\}.\end{aligned}$$

In an exactly similar manner we should obtain

$$df(q) = dq\,.\,f'(q) + dUVq\,.\,\{TVf(q) - TVq\,.\,f'(q)\}.$$

If UVq be constant, *i.e.* if the vector of q preserve a constant direction, then

$$dUVq = 0 \text{ and } \nabla UVq = 0,$$

and in that case we have the two relations

$$\nabla f(q) = \nabla q\,.\,f'(q), \quad df(q) = dq\,.\,f'(q).$$

The equation $dUVq = 0$ requires that $UVq = \text{const.}$, and it is easily established that if q is to be a real quaternion then the equation $\nabla UVq = 0$ also requires the same condition.

4. Instead of the elementary solutions of $\nabla r = 0$ used in my former paper, we might have taken the simpler pair

$$u = y + kx, \quad v = z - jx,$$

in which case our theorem would have taken the form

$$dr = du\,.\,\frac{\partial r}{\partial y} + dv\,.\,\frac{\partial r}{\partial z}.$$

Now, if λ and μ be scalar constants,

$$UV(\lambda u + \mu v) = \frac{\lambda k - \mu j}{\sqrt{\lambda^2 + \mu^2}},$$

and is, therefore, constant. We also have

$$\nabla(\lambda u + \mu v) = 0,$$

and it, therefore, follows by the preceding article that

$$\nabla\,.\,e^{\lambda u + \mu v} = 0.$$

Thus we see that the general integral of the equation $\nabla r = 0$ may be written in the symbolical form

$$r = e^{u\frac{\partial}{\partial\lambda} + v\frac{\partial}{\partial\mu}} f(\lambda, \mu),$$

where

$$f(\lambda, \mu) = A + B\lambda + C\mu + \frac{1}{2!}\{D\lambda^2 + 2E\lambda\mu + F\mu^2\}$$
$$+ \frac{1}{3!}\{G\lambda^3 + 3H\lambda^2\mu + 3K\lambda\mu^2 + L\mu^3\} + \&c.,$$

the coefficients A, B, C, &c., being in general quaternions. The generality of this form will not be affected if after expansion of the exponential symbol and subsequent differentiation we make λ and μ both zero. Hence we obtain

$$r = A + uB + vC + \frac{1}{2!}\{u^2D + (uv + vu)E + v^2F\}$$

$$+ \frac{1}{3!}\{u^3G + (u^2v + uvu + vu^2)H + (uv^2 + vuv + v^2u)K + v^3L\} + \&c.$$

It is important to notice that the quaternion coefficients must always be placed last in the various terms of the series. Graves[1] gave a form similar to this as a quaternion solution of Laplace's Equation, but he seems to have only contemplated the possibility of scalar coefficients, and in consequence his result loses in generality. The importance of introducing quaternion coefficients will be at once seen if we attempt to express the elementary solutions mentioned in my former paper in terms of those used in this. We have

$$-2ix + jy + kz = (y + kx)j + (z - jx)k,$$
$$ix - 2jy + kz = -2(y + kx)j + (z - jx)k,$$
$$ix + jy - 2kz = (y + kx)j - 2(z - jx)k,$$
$$2(x + y + z) + i(y - z) + j(z - x) + k(x - y)$$
$$= (y + kx)(2 + i - k) + (z - jx)(2 - i + j).$$

By means of the equation

$$df(q) = dq \,.\, f'(q)$$

of the preceding article, which holds in the case under consideration, we have

$$dr = \left(du \,.\, \frac{\partial}{\partial\lambda} + dv \,.\, \frac{\partial}{\partial\mu}\right) e^{u\frac{\partial}{\partial\lambda} + v\frac{\partial}{\partial\mu}} f(\lambda, \mu)\,^2$$

$$= du \,.\, e^{u\frac{\partial}{\partial\lambda} + v\frac{\partial}{\partial\mu}} \frac{\partial f}{\partial\lambda} + dv \,.\, e^{u\frac{\partial}{\partial\lambda} + v\frac{\partial}{\partial\mu}} \frac{\partial f}{\partial\mu}$$

$$= du \,.\, U + dv \,.\, V,$$

[1] "On the Solution of the Equation of Laplace's Functions," *Proc. R. I. A.*, VI. 162—171, 186—194. Graves's papers were written with the object of giving the interpretation of a symbolical form obtained by Carmichael, "Laplace's Equation and its Analogues," *Cambridge and Dublin Mathematical Journal*, VII. 126—137.

[2] It is to be understood that after expansion of the exponential symbols and subsequent differentiation, λ and μ are to be made zero.

where U and V are formed from $\partial f/\partial\lambda$ and $\partial f/\partial\mu$ in a similar manner to that in which r is formed from $f(\lambda, \mu)$. Thus we see that if we take u and v as our fundamental solutions of $\nabla r = 0$, the formula discussed in my former paper is very closely analogous to the formula

$$dw = \frac{\partial w}{\partial \xi} d\xi + \frac{\partial w}{\partial \eta} d\eta,$$

which would hold if w were a scalar function of the two variables ξ and η.

Instead of the pair of special solutions that we have here made use of, we might have chosen either of the pairs

$$z + iy \text{ and } x - ky,$$

$$x + jz \text{ and } y - iz.$$

I have investigated the matter and find that, adhering to real quaternions, the most general form that we can take for our pair of special solutions, in order that a third solution may be expressed in terms of them in the simple form of the present article, can by a suitable choice of axes be expressed by

$$y + kx, \quad y \cos\alpha + z \sin\alpha - x(j \sin\alpha - k\cos\alpha).$$

5. The expression for r given in the preceding article is obviously not perfectly general, as it is derived from a series containing only positive integral powers. In the second part of the paper referred to above, Graves gave a method of deriving solutions from scalar series containing negative and fractional powers, but he did not succeed in expressing his results in terms of the two special solutions he made use of. I think that it is highly probable that if we take any two independent special solutions, any other solution can be derived from them; but at the same time it is possible that the said other solution may not be expressible in terms of the two special solutions in the ordinary functional form.

[There is one remark to be made in completion of my former paper. It is there proved that if p and q be any two independent solutions of the equation $\nabla r = 0$, and r any other solution, then there exists a relation of the form

$$dr = dp \,.\, P + dq \,.\, Q,$$

where P and Q do not involve the ratios $dx : dy : dz$. The converse of this is also true: for since the above relation is to be satisfied

for all small variations of the point (x, y, z), it involves the three relations

$$\frac{\partial r}{\partial x} = \frac{\partial p}{\partial x} . P + \frac{\partial q}{\partial x} . Q,$$

$$\frac{\partial r}{\partial y} = \frac{\partial p}{\partial y} . P + \frac{\partial q}{\partial y} . Q,$$

$$\frac{\partial r}{\partial z} = \frac{\partial p}{\partial z} . P + \frac{\partial q}{\partial z} . Q.$$

Hence it follows that

$$\nabla r = \nabla p . P + \nabla q . Q,$$

so that if $\nabla p = 0$ and $\nabla q = 0$, we have $\nabla r = 0$ also.]

[It is to be remarked that a theory similar to the one established in the above paper can be constructed for the equation discussed in Art. 5 of my former paper, by choosing for the elementary solutions the expressions

$$x - it, \quad y - jt, \quad z - kt.$$

April 27th, 1891.]

March 9, 1891.

Dr Lea in the Chair.

H. G. Dawson, M.A., Christ's College, was elected a Fellow of the Society.

The following Communications were made to the Society:

(1) *On the disturbances of the body temperature of the fowl which follow total extirpation of the fore-brain.* By J. George Adami, M.A., M.B., Christ's College, John Lucas Walker Student in Pathology.

[*Abstract; received May* 8, 1891.]

In the course of a series of experiments, undertaken at Paris, upon the development of fever by means of aseptic solutions of the products of bacterial growth it became important to take into consideration how far the various phases of the febrile state depend upon the action of the higher nervous centres; to see whether a typical fever can be induced when the cerebral hemispheres have been removed, and to investigate the terms of relationship between 'the heat-centres' (if these have a local existence) and the altered conditions of heat production and the giving off of heat which obtain during the febrile state.

These questions I do not propose to answer in the present paper, which is but of the nature of a preliminary communication. I can here only state the results of certain first steps towards a resolution of the problems, and note the variations in the temperature of the body, as measured by the thermometer, during the days immediately following upon the removal of the cerebrum.

For these experiments I made use of the fowl. Mammals were out of the question for any prolonged observations, inasmuch as they cease to manifest any vitality whatsoever in the course of but a few hours after the hemispheres have been extirpated; and I employed the fowl rather than the pigeon (which otherwise has many advantages) because with the former it is the more easy under ordinary conditions to induce experimental fever.

In the a-volitional non-sentient state which follows removal of the cerebral hemispheres the fowl, it is well known, may continue for weeks, and it may be months. When however in addition to the hemispheres the optic lobes are to a greater or less extent extirpated, as was purposely the case in my experiments, then this state, it would seem, can last for a much shorter time, the bodily functions ceasing in from one to ten days according to the amount of obliteration of these organs that has been practised. Hence in these experiments it cannot be said that the stage of 'shock' following the operation has definitely been passed: it is impossible to declare that the variations in temperature which I am about to describe are not largely due to the highly irritable condition of the rest of the nervous system brought about by operative interference and removal of the higher centres.

Kept at an equable and moderate external temperature the ordinary fowl exhibits during the day a variation in the body temperature of at most 0·75° C., the mean temperature of well-fed fowls as measured in the rectum being—in winter—about 42·3° C. (108° F.). But after extirpation of the hemispheres and optic lobes—the latter wholly or partially—the temperature variations became very wide, passing from below 35° to above 45° C., and it was a matter of extreme difficulty to prevent, even for a few hours, well-marked ascents or descents of the temperature. Removal, therefore, of so large a portion of the brain had thoroughly disturbed the balance between the thermogenic and thermolytic powers of the organism.

So great had been the disturbance that now the fowls reacted to changes in the external temperature much in the same way as do cold-blooded animals. Placed in a room whose temperature was 22° C. (71·6° F.), and covered carefully with cotton wool the rectal temperature rose rapidly until in those instances in which the rise was unchecked it reached the height of 44·5° C.

(112·0° F.) or more, in one case becoming as much as 45·325° C (113·58° F.). Removal of the cotton wool checked the rise and often induced an actual fall through more than a degree in the course of two hours. With an atmosphere slightly warmer (24° C.) no cotton wool was necessary to bring about an ascent. Transference to a room whose temperature was some degrees below 22° C. led to a rapid lowering of the point to which the mercury rose. Thus in one case, the fowl being placed in a room at 24° C. and covered with wool, the body temperature rose four degrees in five hours, from 40·2° to 44·2° C.; transferred to a room at 18° the animal shewed in less than three hours a fall through 5·6 degrees, to 38·6°; then replaced in the room at 24°, this time without a covering of wool, the temperature rose slowly but steadily until it reached 42·8° at the end of eight hours. I might adduce many other instances to the same effect.

Similarly, in animals whose fore-brain had been extirpated, 15 ccm. of cold water poured into the crop caused a fall of from half a degree to a degree during the course of the succeeding half-hour. This amount of cold water has no effect upon the body temperature of the normal fowl. On the other hand a rich proteid diet in the form of an egg, beaten up and warmed to nearly the temperature of the body, caused invariably a well-marked rise of one to three degrees beginning two to three hours after the animal had been fed. This rise reached its maximum in about six hours, and may be compared to the rise that has been found to occur in the crocodile and, I believe, the snake, after a large meal of animal food.

It may be noted here that while change to a cooler atmosphere caused lowering of the body temperature, this lowering, if the change had not been too considerable, tended to give place eventually to a slow rise. To this extent the reaction differs from what obtains in the cold-blooded animal, and despite the removal of the fore-brain there would seem to be a tendency for the body temperature to be brought back to the normal.

With the temperature liable to such great and constant fluctuations it was extremely difficult to determine the effects of injections of fever-producing substances, as, for example, sterilised bouillon in which the *Vibrio Metschnikovi* had been grown, or to know at what moment these might be made. Nevertheless in the two cases in which such injections were performed under what appeared to be favourable conditions there was so immediate and steady a rise of temperature through two degrees during the succeeding eight hours, that I am led to see in this rise an indication that febrile changes may be induced in the hen deprived of its hemispheres, and, if it be accepted that in the fore-brain lies the main heat regulating mechanism of the

body, that febrile changes may be induced independently of this mechanism.

(2) *On the nature of Supernumerary Appendages in Insects.* By W. BATESON, M.A., St John's College.

[*Abstract.*]

The author exhibited a number of specimens in illustration of this subject.

The evidence related to about 220 recorded cases of extra legs, antennæ, palpi or wings, and particulars were given as to the mode of occurrence of these structures.

Speaking of cases in which the nature of the extra parts could be correctly determined, it was found that the following principles were generally followed:

I. Extra appendages arising from a normal appendage usually contain all parts found in the normal appendage peripherally to the point from which they arise, and never contain parts central to this point.

II. Such appendages are commonly double. The axes of the three appendages then stand in one plane, one being nearer to the normal appendage and one remote from it. In structure and position the nearer limb is the *image* of the normal limb in a mirror perpendicular to the plane in which the limbs stand, while the remoter extra appendage is the image of the nearer one in a remote mirror parallel to the first. Thus if the normal limb is a right limb, the nearer supernumerary is a left and the remoter a right, and *vice versâ*.

An extra appendage sometimes occurs which is apparently a single structure. In all instances in which the matter could be determined, it was found that the apparently single appendage in reality consisted either of two anterior halves or of two posterior halves of a *pair* of appendages conforming to the law stated. Probably therefore no extra appendage is morphologically single.

It was pointed out that these phenomena are important as an indication of the physical nature of bodily symmetry, and in their bearing upon current views of the character of germinal processes.

The author expressed his indebtedness for information, or the loan of specimens, to Messrs H. Gadeau de Kerville, Pennetier, Giard, Kraatz, L. von Heyden, Dale, Mason, Westwood, Waterhouse, N. M. Richardson, Janson, Reitter, &c., and especially to Dr Sharp for much help and advice in examining the specimens.

(3) *On the Orientation of* Sacculina. By Theo. T. Groom, B.A., St John's College.

[*Abstract; received March* 10, 1891.]

In comparing the larval stages of various members of the group Cirripedia I have found it necessary to come to some independent conclusion as to the relation of the adult animals in the several groups to one another. One of the questions raised concerns the morphology of the Rhizocephala.

The Rhizocephala are, as is well known, a group of small animals parasitic on Decapod Crustacea.

It is only of late years that we have obtained anything like an accurate knowledge of the structure and life-history of any member of this group; but since the appearance of Delage's classical paper[1] on the development of *Sacculina* we have had a fair knowledge of one species.

Two views as to the orientation of *Sacculina* have been maintained, the earlier by Kossmann[2], and the later by Delage.

In order to determine the orientation of *Sacculina* it will be necessary to briefly compare the structure of the adult with that of a typical Thoracic Cirripede such as *Lepas* or *Pollicipes.*

In both, the whole structure indicates a primitive bilateral symmetry on each side of a median plane. At one end of the body is a *peduncle* or more or less elongated stalk by which the animal is attached: this in *Sacculina*, in accordance with its parasitic mode of life, gives off rootlets for the absorption of nutriment. The body gives off in the median plane in close relation with the peduncle at one point the *mantle*, the histology of which presents considerable similarity of a special character in the two forms, as seen in the accounts of Delage in *Sacculina* and of Koehler[3] and Nussbaum[4] in *Lepas*, *Pollicipes*, etc. The differences in histology between the two are evidently closely connected with the absence in *Sacculina* of the strong calcareous plates of *Lepas* and the consequent predominance of muscular and connective tissue elements, a result of the different modes of life of the two forms. The mantle surrounds the body on all sides, forming the *mantle-cavity* closed except at the end remote from the peduncle, where the *mantle-opening* leads to the exterior. The mantle-cavity has in both the same function of retaining and probably of aerating the eggs. The mantle is attached to the body by a band of tissue not distinct from the peduncle in the adult *Lepas* (although

[1] Evolution de la Sacculine. *Archives de Zool. Exp. et. Gén.*, 2 Série, Tome 2, 1884.

[2] Suctoria und Lepadidae. *Arbeiten a. d. zoolog. zoot. Institut in Würzburg*, 1, 1874.

[3] Recherches sur l'organisation des Cirrhipèdes. *Archives de Biologie*, IX, 1889.

[4] Anatomische Studien an Californischen Cirripedien. Bonn, 1890.

definitely recognisable in the larva) but forming a membrane in *Sacculina* running from the peduncle to the mantle-opening and termed by Delage the *mesentery*. Symmetrically situated on each side of the median plane are the *oviducts* opening into the mantle-cavity by the two *female genital pores*[1]. The relation of the oviduct to the ovaries is similar in both cases, though in consequence of the different situation of the ovaries (due perhaps to the special mode of nutrition in *Sacculina*) the relative lengths are very different: in both, the lower part of the oviduct is expanded to form a chamber[2], the glandular walls of which, simple in *Lepas*, but branched in *Sacculina*, constitute the *Kittdrüsen* of the Germans and "*Glandes cementaires*" of Delage (as Giard[3] has pointed out and Delage admits, these glands have no relation with the cement-glands proper of Cirripedia). The function of these glands in both groups is to produce the peculiar sac or cocoon[4] in which the ova are enclosed when lying in the mantle-cavity as the *ovigerous lamellae* of Darwin[5].

The openings of the vasa deferentia (also symmetrical) in Cirripedes are less constant in position than the other organs mentioned, and I wish to reserve all mention of them for a future occasion.

The last structure to be compared in the two forms is the nervous system. This in Lepads consists of a supra-œsophageal, a large sub-œsophageal and a series of posterior ganglia. In *Sacculina* a single ganglion is present, corresponding probably in position with the sub-œsophageal ganglia of Lepads or Balanids, both being near the female genital apertures.

Now both Kossmann and Delage are agreed that the plane of symmetry passing through the mesentery in *Sacculina* is the median vertical plane. Kossmann places the peduncle in front, the mantle-opening behind and the mesentery on the dorsal line. Delage making the mesentery ventral and the mantle-opening posterior gives *Sacculina* a position diametrically opposite to that of Kossmann. He bases his view solely on the situation and origin of the ganglion. Since, he argues, this arises on the ventral side in articulate animals, the neighbourhood of the edge of the mesentery close to which the ganglion originates must also in *Sacculina* be ventral; in other words the mesentery is ventral; and since the nervous centre in Cirripedes is situated in the region of the thorax, the ganglion of *Sacculina* must indicate the

[1] Krohn, Wiegmann's *Archiv. f. Naturgeschichte*, 1859. Hoek, Challenger Report, Vol. x., Cirripedia, Anatomical Part, 1884. Nussbaum, *loc. cit.*

[2] Hoek, Nussbaum, *loc. cit.*

[3] Sur l'orientation de Sacculina Carcini. *Comptes Rendus*, 102.

[4] Hoek, Nussbaum, *loc. cit.*

[5] A monograph of the Cirripedia, Lepadidae and Balanidae, Ray Society, 1851, 1854.

point where the thorax would be if present; *i.e.* the posterior end of the animal.

I must admit, however, that it seems to me quite impossible to determine the orientation from one point alone, and that given the ganglion were situated at the posterior end of the body on the ventral side, I fail to see why the side on which the mesentery is situated cannot be equally dorsal or ventral. We need more than one point to determine the orientation of any animal, and this it seems to me is given in the present case by the comparison of other structures. Delage, however, rejects the evidence furnished by the other organs, and bases this rejection on the embryology of *Sacculina*, of which he has given so interesting an account. He finds that upon the fixation of the Cypris-form the young *Sacculina* loses all its organs (carapace, appendages, etc.) and becomes reduced to a mass of embryonic cells from which all the organs of the adult *Sacculina* develop *de novo.* He concludes from this that not only are the organs of the adult morphologically different from those of the Cypris-stage, but that there is no necessary agreement of any one of the surfaces of the adult with any one of the larva, the former being determined wholly by the relation of the parasite to the host.

The development is certainly remarkable, but I think there is no reason to doubt as Giard[1] has done the general correctness of Delage's account. The development of organs *de novo* occurs, however, in other forms and the case in point seems to me hardly more marvellous than the re-development of the three posterior pairs of maxillipedes in *Squilla* after they have been once lost[2], or the similar reproduction of the last two pairs of thoracic appendages in *Sergestes*[3], and other analogous cases: yet few would venture to doubt the homology of these appendages with the corresponding ones in allied forms[4].

The tendency of late years has been, I think, to admit that a very considerable amount of modification of the ancestral development may take place, and that we must be exceedingly cautious before admitting any case of ontogeny as presenting a truthful representation of the phylogeny. The comparison of the structure of the adults will in many cases be of greater service.

It appears to me, therefore, that the cumulative evidence of

[1] Giard, *loc. cit.*

[2] Balfour, A Treatise on Comparative Embryology, vol. I., 1880.

[3] Balfour, *op. cit.*

[4] I must point out that the nervous centre is one of those organs which, according to Delage, arises *de novo*, and that if such organs are not to be regarded as homologous with those of the larva, Delage is hardly justified in determining the orientation of *Sacculina* by means of the position of the ganglion, since it can hardly be doubted that the nervous system of the *Sacculina* Cypris-form is the same thing as that of the Cypris-form and adult of Thoracic Cirripedes.

the organs compared in *Lepas* and *Sacculina* speaks very strongly in favour of their homology in the two, and if this be the case there can be little doubt as to the true orientation of *Sacculina*, and that though the supposed development on which Kossmann partly based his case has been shewn by Delage not to take place it will be necessary to return to the older view based upon less complete information, that the mesentery is dorsal and the mantle-opening posterior.

There is one more point to which I wish to draw attention. The great similarity in structure and relation of the oviduct in *Lepads* and in *Sacculina* makes it exceedingly probable that these organs are truly homologous. Now in all Cirripedes in which the position of the genital pores has been accurately determined they are very constantly situated at the base of the first of the six pairs of cirri[1]. In all probability then the portion of the body on which the two oviducts open is thoracic. This conclusion agrees well with the position of the ganglion which corresponds closely with that of the largest ganglion of the Thoracica. I would suggest then that when fixation of the Cypris-form takes place a great reduction in size of the thorax occurs at the same time as the moult accompanying that fixation, and that the whole of the thorax and abdomen is not cast off in the way Delage supposes, but that a portion, small perhaps, but morphologically representing them is left attached to the head. This would indicate a conception of the Rhizocephala differing considerably from that proposed by Delage who considers the adult *Sacculina* to represent the head alone of other Crustacea. The visceral portion of the parasite would consist of head and thorax fused together; the latter, judging from the position of the female genital pores, occupying as much as one-quarter of the whole. The carapace attached to the dorsal side of the head in *Lepas* and in the Cypris stage of *Sacculina* must then be regarded as fused to the dorsal side of the thorax as well as the head, in a manner analogous to that of the higher Crustacea.

(4) *Some experiments on blood-clotting.* By ALBERT S. GRÜNBAUM, Gonville and Caius College.

[*Abstract; received March* 3, 1891.]

In the 'Centralblatt für Physiologie' of 1888, p. 263, there is a paper by K. Bohr on the respiratory changes due to injection of peptone and leech-extract, and in this paper he incidentally

[1] I have lately been able to supply the only missing link in the chain of evidence which points to Darwin's "Acoustic sacs" as marking the termination of the oviduct: the eggs in one specimen of *Lepas* were found actually issuing from the orifice of the sac.

mentions that if, in the rabbit, the cœliac and superior mesenteric arteries be tied, the blood, if collected after about four hours, will not clot for about two hours. He intimates his intention of enquiring further into this matter, since his assertion is based on two experiments only; but he has as yet written nothing more on the subject.

In April, 1890, I commenced repeating his experiments, but I have not yet had time to investigate the matter completely. However, in six experiments in which the cœliac and superior mesenteric arteries were ligatured, the blood collected always clotted in less than two hours: in fact, generally under ten minutes. In one case only were twenty minutes required before the clotting became evident. It is true that the time was slightly lengthened, since, normally, clotting takes place in 3–5 minutes; and the clot was decidedly not so firm as usual. The experiments were performed as Bohr describes, except that in the first two, the abdomen was opened in the middle line, instead of at the side. After four hours, the animal being under chloroform during the whole time, the blood was collected from the carotid. Bohr states that his animals were apparently as well at that time as at the beginning: I always found the stomach partially self-digested and the intestines becoming gangrenous, and there was a great fall of temperature.

The occlusion of the arteries by the ligature was always tested by injecting the animal after death, and, with one exception in which the cœliac artery let the injection through, was found to be complete. Thus it appears that some slight though evident effect on the clotting may be produced by stopping the circulation through the stomach and intestines, but it is generally not as marked as in Bohr's two experiments.

May 4, 1891.

PROF. G. H. DARWIN, PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:

(1) *The most general type of electrical waves in dielectric media that is consistent with ascertained laws.* By J. LARMOR, M.A., St John's College.

It was explained that Maxwell's hypothesis of complete circuits reduces the whole of electrodynamics to the ascertained Ampère-Faraday-Neumann laws for such circuits, as it completely defines the character of the electrostatic polarisation which must be postulated as a part of the theory. The question as to how far the theory of electrodynamics may depart from this simple form

when the circuits are not assumed to be complete was then examined mathematically on the lines of v. Helmholtz's investigations. It was shown that waves of transverse displacement will always be propagated in a dielectric, irrespective of what hypothesis is assumed as to the law of the mutual action of incomplete currents, whether that adopted by v. Helmholtz or the still more general one which is formally possible. But it was also shown that, if the velocity of these waves in a non-magnetic dielectric is equal to the inverse square root of the specific inductive capacity, the currents are necessarily complete. The experimental evidence is strongly in favour of this relation, and in so far constitutes a demonstration of Maxwell's theory of electrodynamic action and its mode of propagation in stationary media[1].

(2) *A mechanical representation of a vibrating electrical system, and its radiation.* By J. Larmor, M.A., St John's College.

The propagation of undulations of electric polarisation in a dielectric is exactly similar to the propagation of elastic waves in a solid medium, which must be absolutely incompressible if we follow Maxwell's scheme, but may transmit waves of condensation as well as shearing waves if we admit the more general scheme developed by von Helmholtz.

The propagation of electrical actions in a medium which is a conductor of ordinary type (so that the number which expresses its inductive capacity in C.G.S. electromagnetic measure is very small, while that which expresses its specific resistance is large) follows the same law as the diffusion of heat in a conducting medium, with the proviso that the thermal diffusivity is to be proportional to the reciprocal of the electric conductivity. For this reason rapidly alternating disturbances in the medium surrounding a conducting mass are only transmitted skin deep into the conductor, the depth of this skin diminishing with increasing rapidity of the alternations and increasing conductivity, in the same manner as in the corresponding question of the propagation beneath the surface of the ground of the daily and annual alternations of temperature. For this reason also a sheet of conducting metal acts as a screen against alternating electrodynamic influence.

In development of this elastic solid analogy it has recently been explained by Sir W. Thomson[2] that the magnetic induction due to a steady electric current traversing a circuital channel in the medium is represented by twice the vorticity of the elastic strain when a longitudinal force of amount represented by the current is applied to the portion of the medium which coincides with this channel.

[1] See *Proc. Roy. Soc.* 1891, Vol. XLIX., p. 521.
[2] *Math. and Phys. Papers*, Vol. III. p. 451.

It is also a well-known relation, and forms in many respects the simplest and most symmetrical mathematical specification of the connexions of the electric field on Maxwell's scheme, that the magnetic induction (abc) is represented by the vorticity of the electric force (PQR), and the electric force by the vorticity of the magnetic induction, according to the equations

$$\frac{dQ}{dz} - \frac{dR}{dy} = \frac{da}{dt}, \ldots, \ldots,$$

$$\frac{db}{dz} - \frac{dc}{dy} = -4\pi\mu\left(K\frac{d}{dt} + \sigma\right)P, \ldots, \ldots.$$

In the dielectric, σ is zero, and the electric displacement (fgh) is proportional to the electric force according to the relation

$$(f, g, h) = \frac{K}{4\pi}(P, Q, R).$$

Thus the vorticity of the electric displacement is $\frac{K}{8\pi}\frac{d}{dt}$ of the magnetic induction; and the time integrals of (fgh) represent on Sir W. Thomson's analogy the displacements of the elastic medium multiplied by the factor $\frac{K}{4\pi}$.

Or we may, as in the following sections, take the electric displacement to represent the actual displacement of the elastic medium, and then the magnetic induction will be equal to the time integral of its vorticity multiplied by $\frac{8\pi}{K}$; and the electric current will be equal to the time integral of the impressed forcive multiplied by $\frac{4\pi}{K}$. This forcive must on Maxwell's scheme be circuital, that is, circulating in ideal channels in the manner of the velocity system of an incompressible fluid.

These considerations thus present a mechanical view of the electric propagation in dielectrics, with the exception that the current on the wire must be imitated by an applied forcive of some kind. If the media are magnetic, differences in rigidity or of density must be introduced between them.

For an electrical vibrator we can however complete the mechanical analogy, provided the wave-length of the undulations is not smaller than a few centimetres in the case when the vibrator is made of an ordinary conducting metal. For experimental types of Hertzian vibrators the analogy will therefore be practically exact.

To do this we have to examine the conditions as regards electric displacement that must be satisfied at the surface of a conductor.

The displacement is proportional to the electric force, which consists of a kinetic and a static part. The kinetic part is

$$-\frac{d}{dt}(F,\ G,\ H),$$

where $$\nabla^2(F,\ G,\ H) = -4\pi\mu\,(u,\ v,\ w)$$

because FGH is the vector potential of the current distribution; it is therefore continuous everywhere as the potentials of all volume or surface distributions must be. The static part

$$-\left(\frac{d}{dx},\ \frac{d}{dy},\ \frac{d}{dz}\right)V,$$

can only be due to a surface density s on the conducting surface; therefore its components along the surface must be continuous when we cross it, while the discontinuity in the component normal to the surface must be equal to $4\pi s$. The tangential components of the total electric force are therefore continuous.

Now it is known that for rapid alternations the currents are induced only in the skin of the conductor; the vibrations of the system and its free periods are in the limit quite independent of the conductivity and of the nature of the interior parts of the conductor: they are practically the same as if the conductivity were infinite, and there is no sensible decay owing to any degradation into heat.

We require the proper boundary conditions to express these facts.

The currents in the conductors may be treated as surface sheets inside which the electric force is zero.

Outside the sheet therefore the components of the electric force along the surface must be zero also; and therefore the tangential electric displacements must be zero.

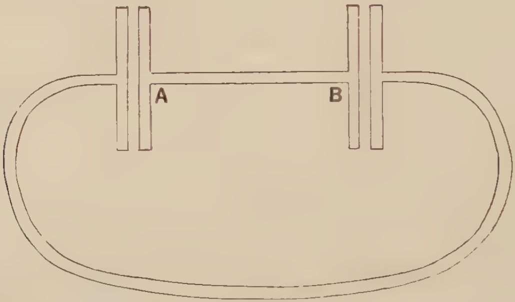
The components of the electric force normal to the surface will differ on the two sides of it by an amount determined by the surface density s.

In the elastic solid analogy we may therefore consider the conductors to be cavities in the solid, provided we confer infinite rigidity on the skin of each cavity so that no point of it can have any tangential displacement. If we assume that the solid is incompressible, its vibrations are completely determined by these conditions; the component of the displacement normal to the surface must naturally adjust itself in such manner that the condensation remains always null. This will be the analogue of an electric surface density on the conductor which adjusts itself instantaneously to the equilibrium value corresponding to the actual phase of the disturbance in the dielectric, according to

the equation $\nabla^2 V = 0$ which of course indicates infinite velocity of propagation or adjustment of disturbances.

The medium being incompressible, the total volume of all the cavities will remain the same. We may put this necessary property of the motion in evidence by supposing each cavity to be filled with incompressible fluid. The displacement of this fluid at the surface will then be continuous with the displacement of the solid and will represent the electric surface density. The inertia of the fluid must not sensibly interfere with the adjustment of the normal displacement; so that if the rigidity of the solid is taken to be finite, the fluid must be of negligible density. But it is to be borne in mind that, as the fluid represents a conductor, the motion of the fluid does not represent electrostatic displacement except on the surface.

We may thus represent the circumstances of an electric vibrator by the annexed diagram.



Two condensers A, B, are represented, with their inner coatings connected by a conducting wire in which the spark gap required for the production of the initial disturbance is usually situated; and their outer coatings are connected by another wire which may be to earth. Each conducting system forms a cavity in the elastic dielectric, which may be considered to be filled with incompressible massless liquid; in this case there are two such cavities with pairs of plane faces opposed to each other.

The disturbance may be supposed to be originated by getting an excess of liquid into the inner coating of the condenser A; that involves pushing away the plate of dielectric between the two coatings of this condenser, and therefore removing an equal amount of liquid from the outer coating. For the parts of the conducting surfaces other than the opposed faces are all backed up by thick masses of dielectric, which will not yield sensibly as compared with the thin plates of dielectric which belong to the condensers. The greater mobility of the latter accounts both for the store of energy that the condenser can acquire, and for the equality of the charges of its two coatings.

When the system is disturbed the liquid will sway backwards and forwards between the condenser coatings in a way which gives a very real representation of the actual electric oscillation.

The only element in this representation that is not easily realisable is the skin, of rigidity great compared with that of the solid; if however the solid were a jelly the skin would be naturally provided by supposing the system represented in the diagram to be constructed of very flexible sheet metal.

The greater the capacity of the condensers compared with the section of the connecting wire the longer is the period of the graver vibrations; thus illustrating the dependence of the period on the capacity and self-induction of the vibrator. There are also overtones in which the pulse of normal displacement is almost confined to the connecting wire, the greater mobility at its ends making those points approximately nodes; and their nodal character will be the more prominent the greater the capacity per unit area at the points where they are attached to the condensers. For these overtones the half wave-length is thus a sub-multiple of the length of the connecting wire.

Owing to the small surface of the connecting wire these overtones would not have much chance of being communicated to the surrounding medium, except by reason of the general principle which requires that the shorter the period for given dimensions of the vibrator the more of the vibrational energy travels outwards into the medium, and that in a ratio which increases very rapidly with increasing frequency. In the Hertzian oscillator the condensers are replaced by large metallic plates, and it seems clear that the waves that are the chief subject of experiment issue from the large surfaces afforded by these plates, and so belong to the lower periods of the vibrator.

The waves of a Hertzian vibrator are therefore radiated from the plates of the vibrator: to obtain considerable radiation to a distance it is essential that the dimensions of a plate should be considerable compared with the length of a wave of the radiation. This condition would not be fulfilled if the plates were replaced by condensers, for though the energy of the vibration would thereby be increased, its wave-length for the fundamental periods would be lengthened, and the radiation, therefore, very much diminished. Condensers would be allowable instead of plates when the disturbance is guided along conducting wires, but as a rule they would give only slight undulations in the dielectric at a distance from conductors.

The plates are thus like two poles radiating in opposite phases: and the general equations given by Kirchhoff to represent this type of motion are those used by Hertz in his discussion of the general character of the radiation at a distance from the vibrator.

[The different types of vibration that may theoretically exist would therefore appear to be as follows. A reciprocating flow may be set up between the plates, along the connecting wires; if the capacities are at all considerable its period will be comparatively slow, being calculable from capacity of condenser and self-induction of wire alone, because there will be involved very little disturbance in the dielectric except the static value of the displacement at each instant, and there will be no sensible amount of radiation. An oscillation of the dielectric to and fro between the two plates may be set up, corresponding to a very small period, the wave-length being twice the distance between the plates in the case in the diagram, when the earth connexion is good; this will not be sensibly affected by the presence or nature of the connecting wire, and it will involve rapid decay by radiation, but it will probably be very difficult to excite to any sensible amount. And there may be superficial dielectric waves running along the connecting wire, of period about the same as the preceding when the wire is straight.

For the linear type of vibrator consisting of two equal cylinders with a spark gap between them, and no capacities at the ends, the wave-length would be the length of the vibrator simply, as from the mode of excitation its two halves would always be in opposite phases.]

The nature of the approximations involved in this method of representation will be sufficiently put in evidence by the following investigation of the circumstances of the reflexion of a system of waves from a metallic plate. It will be found that for wave-lengths of a centimetre or more the reflexion of all ordinary metallic plates is sensibly perfect, and involves an acceleration of phase of half a wave-length. For smaller wave-lengths, corresponding to those of light waves, the circumstances of the reflexion are more complicated.

We shall proceed on Maxwell's theory, which postulates the non-existence of condensational effects: no other theory can make the velocity of propagation of waves of transverse displacement inversely proportional to the square root of the specific inductive capacity of the medium[1].

The first point is to assume precise definitions of the quantities that enter into the equations. With the usual notation

$$c=\frac{dG}{dx}-\frac{dF}{dy},\ a=\ldots,\ b=\ldots,$$

so that, if J denote

$$\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz},$$

[1] See *Proc. Roy. Soc.* 1891, Vol. XLIX. p. 521.

we have

$$\nabla^2 H - \frac{dJ}{dx} = \frac{db}{dx} - \frac{da}{dy}$$
$$= -4\pi\mu w.$$

Hence *defining* FGH by the formula

$$F,\ G,\ H = \int \frac{d\tau}{r} \mu\,(u,\ v,\ w),$$

we annul J, since

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0,$$

owing to the absence of condensation.

The components of electric force are

$$P = -\frac{dF}{dt} - \frac{dV}{dx},\ Q = \ldots,\ R = \ldots,$$

where V is a function whose presence is necessitated by the fact that the currents must be circuital; it must enter in this form in order that it may produce no result on integration round a circuit, and it is completely determined by the conditions of any special problem. The fact that V must be a single-valued function so as to give a null result on integrating round a circuit shows that it may be expressed as the potential of an electrostatic distribution; thus its introduction into the equations is accounted for consistently with ordinary electrical ideas.

In a dielectric the total current uvw is the displacement current in the dielectric, so that

$$(u,\ v,\ w) = \frac{K}{4\pi}\frac{d}{dt}(P,\ Q,\ R).$$

In a conductor the displacement current is evanescent compared with the current conducted according to Ohm's law, so that

$$\sigma\,(u,\ v,\ w) = (P,\ Q,\ R),$$

where σ is the specific resistance of the medium.

Also, by definition above,

$$\nabla^2 (F,\ G,\ H) = -4\pi\mu\,(u,\ v,\ w).$$

Thus in a dielectric the equations of propagation of the vector FGH are of the type

$$\frac{1}{\mu K}\nabla^2 F = \frac{d^2 F}{dt^2} + \frac{d^2 V}{dx\,dt};$$

in a conductor, they are of the type

$$\frac{\sigma}{4\pi\mu}\nabla^2 F = \frac{dF}{dt} + \frac{dV}{dx}.$$

In the former case we derive at once

$$\frac{d}{dt}\nabla^2 V = 0,$$

which merely expresses the fact that there can be no accumulation of volume density in the dielectric, on account of the assumed non-condensational character of the phenomena.

In the latter case also

$$\nabla^2 V = 0.$$

Thus, when the dielectric has no initial charge, the charge is throughout confined to the interfaces separating media of different quality, such as the surfaces of conductors.

It will form a sufficiently general case for our purpose if we consider the reflexion of a train of plane polarised waves at a plane metallic surface. The axis of z may be taken normal to the surface and the axis of x along it in train with the waves. If the vibration of the vector potential is in the plane of incidence, we have

$$\frac{dF}{dx} + \frac{dH}{dz} = 0, \quad G = 0,$$

so that the variables are reduced to dependence on a single function χ by the substitution

$$F = \frac{d\chi}{dz}, \quad H = -\frac{d\chi}{dx}.$$

The important case is when everything is periodic, and so involves the factor *exp* $(-\iota pt)$. We have then

$$F = F' - \frac{1}{\iota p}\frac{dV}{dx}, \quad H = H' - \frac{1}{\iota p}\frac{dV}{dz},$$

where
$$F' = \frac{d\chi'}{dz}, \quad H' = -\frac{d\chi'}{dx};$$

and χ' is determined by the equations

$$\frac{1}{\mu_1 K_1}\nabla^2\chi' = \frac{d^2\chi'}{dt^2}$$

in the dielectric, and

$$\frac{\sigma_2}{4\pi\mu_2}\nabla^2\chi' = \frac{d\chi'}{dt}$$

in the conductor.

At the interface F is continuous, so that $\frac{d\chi}{dz}$ is continuous; and H is continuous, so that

$$-\frac{d\chi'}{dx} - \frac{1}{\iota p}\frac{dV}{dz} \text{ is continuous;}$$

these follow from the definition of F, G, H as the potentials of volume distributions.

Further, the component magnetic induction along the normal must be continuous across the interface, which being zero it is; and the magnetic force along the interface must be continuous, so that $\frac{1}{\mu}\left(\frac{dF}{dz} - \frac{dH}{dx}\right)$ is continuous across the surface, that is

$$\frac{1}{\mu}\nabla^2\chi' \text{ is continuous,}$$

or by the equations for χ',

$$K_1\frac{d^2\chi_1'}{dt^2} = \frac{4\pi}{\sigma_2}\frac{d\chi_2'}{dt},$$

or finally, for this special case of harmonic waves

$$-\iota p K_1\chi_1' = \frac{4\pi}{\sigma_2}\chi_2'.$$

Let

$$\chi_1' = A_1 \, exp \, \iota(lx - n_1z - pt) + B_1 \, exp \, \iota(lx + n_1z - pt),$$
$$\chi_2' = A_2 \, exp \, \iota(lx - n_2z - pt),$$

A_1, B_1 thus representing the coefficients of the incident and reflected waves, and A_2 that of the surface-wave in the conductor for which n_2 must in the result be complex.

The value of l must be the same for all these waves, as their traces on the surface must move along it with the same velocity.

The differential equations satisfied by χ' give

$$\frac{1}{\mu_1 K_1}(l^2 + n_1^2) = p^2,$$

$$\frac{\sigma_2}{4\pi\mu_2}(l^2 + n_2^2) = \iota p.$$

The first of these gives the velocity of the dielectric wave to be $(\mu_1 K_1)^{-\frac{1}{2}}$, as it ought to be. The second gives the penetration of the surface-wave into the conductor by the equation

$$n_2^2 = l^2 - \frac{4\pi\mu_2 p}{\sigma_2}\iota.$$

If i denote the angle of incidence of the waves, λ their length, and v their velocity, we must have

$$ln + n_1 z - pt \equiv \frac{2\pi}{\lambda}(x \sin i + z \cos i - vt).$$

Thus
$$\frac{p}{l^2} = \frac{v}{\sin i} \frac{\lambda}{2\pi \sin i};$$

where $v = (\mu_1 K_1)^{-\frac{1}{2}}$ is of the order 3.10^{10} C.G.S., and for light waves λ is of the order 10^{-4}, and for copper σ_2 is about 1600.

Hence for all realisable wave-lengths, long or short,

$$n_2 = \left(\frac{4\pi\mu_2 p}{\sigma_2}\right)^{\frac{1}{2}} \frac{1-\iota}{2}$$

$$= \pi \left(\frac{2\mu_2 v}{\sigma_2 \lambda}\right)^{\frac{1}{2}} (1-\iota),$$

which for light waves is of the order 10^5, and is very large for all realisable waves, unless for media of slight conductivity compared with metals. The amplitude of the surface wave in the conductor is reduced in the ratio e^{-1} at a depth n_2^{-1}; this wave is therefore absolutely superficial, and is fully developed even in a very thin sheet of metal.

The surface conditions give

$$n_1(A_1 - B_1) = n_2 A_2,$$

$$l(A_1 + B_1 - A_2) = -\frac{\iota}{p} 4\pi\sigma_0$$

$$-\iota p K_1 (A_1 + B_1) = \frac{4\pi}{\sigma_2} A_2,$$

of which the first and third equations determine the amplitudes of the waves produced by the reflexion of A_1, and the second determines the surface density

$$\sigma_0 \, exp \, \iota(lx - pt)$$

of the superficial electric charge.

Hence
$$\left(\frac{n_2}{n_1} + \frac{4\pi}{p\sigma_2 K_1}\iota\right) A_2 = 2A_1,$$

i.e.
$$A_2 = 2 \frac{\frac{n_2}{n_1} - \frac{4\pi}{p\sigma_2 K_1}\iota}{\left(\frac{n_2}{n_1}\right)^2 + \left(\frac{4\pi}{p\sigma_2 K_1}\right)^2} A_1,$$

and
$$B_1 = A_1 - \frac{n_2}{n_1} A_2.$$

The relative magnitudes of the two terms in the numerator of A_2 have to be estimated; the terms to be compared are of the orders

$$\left(\frac{v\lambda}{\sigma_2}\right)^{\frac{1}{2}}, \text{ and } \frac{\lambda}{v\sigma_2 K_1} \text{ or } \frac{\lambda v}{\sigma_2},$$

that is $10^4\lambda^{\frac{1}{2}}$ and $10^7\lambda$, roughly.

For light waves, λ is of the order 10^{-4}, so that neither of these terms can be neglected compared with the other; and the completion of the solution will correspond to the somewhat complicated circumstances of the metallic reflexion of light.

But for waves comparable to a centimetre in length, or longer, the second term is negligible; and then

$$A_2 = 2\left(\frac{n_2}{n_1}\right)^{-1} A_1,$$

and
$$B_1 = -A_1.$$

The wave is therefore reflected clean but with opposite phase. And the value of n_2 given above shews that the longer the waves the slighter is their penetration into the conductor; so that even for a curved surface like that of a wire this solution has an application.

The meaning of this approximation is that the first surface condition, in the form it assumes when n_2 is very great, supplies all the necessary data for the motion in the dielectric. That surface condition is equivalent to the statement that F is zero in the dielectric along the surface, and therefore so is the tangential displacement.

Thus the tangential surface conditions suffice in this case to give a full account of the dielectric phenomena, the normal conditions being simply left to take care of themselves.

The function V by which the adjustment is made in the conductor to obtain the normal displacement at the surface which shall satisfy the condition of zero condensation is derived from the characteristic equation $\nabla^2 V = 0$, the same equation as that for the pressure in a homogeneous massless fluid; it of course indicates instantaneous adjustment to an equilibrium value throughout the volume.

(3) *On the Theory of Discontinuous Fluid Motions in two dimensions.* By A. E. H. Love, M.A., St John's College.

This paper contains an exposition of a modification of Mr Michell's method published in *Phil. Trans. R. S., A.* 1890. The motion of the fluid is supposed to take place in the plane of a complex variable z, and to be given by means of a velocity-potential

ϕ and a stream-function ψ, so that $\phi + \iota\psi$ or w is a function of z. The object of the theory is to show how in any given problem the functional relation between w and z can be discovered. For this purpose we consider two functions ζ and Ω such that

$$\zeta = dz/dw \text{ and } \Omega = \log \zeta,$$

and it is well known that Ω is a complex quantity, whose real part is the logarithm of the reciprocal of the velocity of the fluid at the point z, and whose imaginary part is the angle which the direction of this velocity makes with the real axis in the z plane. In the kind of problems to which the method is applicable the region of the z plane within which the motion takes place is bounded partly by fixed straight lines and partly by free stream-lines. Along the fixed boundaries the direction of the velocity is given so that the corresponding parts of the boundary in the Ω plane are lines parallel to the real axis. Along the free stream-lines the velocity is a given constant, so that the corresponding parts of the boundary in the Ω plane are parts of a straight line parallel to the imaginary axis. Hence the boundary in the Ω plane is a polygon which we know how to draw. In like manner the boundary in the w plane, consisting of parts of straight lines parallel to the real axis (ψ = const.,) is a polygon which we know how to draw. If now we take an auxiliary complex variable u the polygons in the Ω and w planes can be conformably represented on the half-plane for which the imaginary part of u is positive. The required transformations are given by the theory of Schwarz and Christoffel, and we are thus in a position to write down two relations

$$\frac{dw}{du} = f_1(u),$$

$$\frac{d\Omega}{du} = f_2(u),$$

from which
$$\frac{dz}{du} = Cf_1(u)\, e^{\int f_2(u)\, du}.$$

The roots and poles of the function $f_1(u)$ are values arbitrarily assumed to correspond to the corners of the polygon in the w plane, and the roots, poles and critical points of the function $f_2(u)$ are in like manner values arbitrarily assumed to correspond to the corners of the polygon in the Ω plane. Of these values three may be arbitrarily fixed, then the rest can be determined. The determination is made by integrating the equation connecting z and u. If the limits of integration correspond to two critical points of the function Ω which lie on the part of the u line that corresponds to a fixed boundary we shall obtain an expression in terms of our assumed arbitrary constants for one of the dimensions of the fixed

boundary. In this way there will arise sufficient relations to determine all the arbitrary constants. If we integrate the equation connecting z and u for values of u that correspond to a free stream-line we shall find for z a complex expression, so that the co-ordinates x and y of any point on the free stream-line will be given functions of a real parameter u.

Mr Michell's theory rests on the properties of the function which is the real part of Ω. He shows how to determine this function in terms of u by considering an analogous electrical problem. When it is known he deduces from it the differential relation between z and u that would be found by the method of this paper.

After explaining the method I give solutions of the following problems:

(i) Mr Michell's problem of the escape of a jet from a tank.

(ii) The flow of liquid against a disc with an elevated rim.

(iii) The impact of a jet against a finite lamina.

(iv) The resistance offered by a plane obstacle placed in a canal of finite width.

(v) The flow of liquid past a pier projecting obliquely.

1. The theory of discontinuous fluid motions in two dimensions, as at present developed, rests essentially on two particular propositions to which we proceed.

Prop. I. The motion being supposed to take place in the plane of the complex variable z, and to be given by means of a velocity-potential ϕ or a stream-function ψ, so that

$$w = \phi + \iota\psi = f(x + \iota y) = f(z),$$

the quantity dz/dw, which we call ζ, is a complex variable whose modulus is the reciprocal of the velocity, and whose argument is the direction of the velocity of the fluid at the point z; and the quantity $\log \zeta$, which we call Ω, is a complex variable whose real part is the logarithm of the reciprocal of the velocity and imaginary part is $\sqrt{(-1)}$ multiplied by the angle the direction of the velocity makes with the real axis in the z plane.

To prove this observe that if u, v be the velocities at any point parallel to x, y, then

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial \phi}{\partial x} + \iota \frac{\partial \psi}{\partial x} = u - \iota v,$$

so that
$$\zeta = \frac{dz}{dw} = \frac{u + \iota v}{u^2 + v^2} = \frac{u + \iota v}{q^2} \text{ say},$$

and
$$\Omega = \log \zeta = \log \frac{u + \iota v}{q^2} = \log \frac{1}{q} + \iota\theta \quad \text{.................(A)},$$

where θ is such that $\cos\theta = u/q$ and $\sin\theta = v/q$.

Prop. II. It is possible to find a transformation by means of a relation between two complex variables Z and Z' by which any given polygon in the Z' plane can be transformed into the real axis in the Z plane, and points within the polygon in the Z' plane correspond singly to points in the Z plane whose imaginary part is positive.

This transformation and the conformable representation of the polygon upon the half-plane, which it involves, have been investigated by Schwarz and Christoffel, and it can be shown that the relation between Z and Z' is given by the equation

$$\frac{dZ'}{dZ} = A \prod_r (Z - X_r)^{\alpha_r/\pi - 1} \quad \ldots\ldots\ldots\ldots\ldots\text{(B)},$$

where A is a constant and X_r is the point on the real axis in the Z plane which corresponds to the internal angle α_r of the polygon in the Z' plane.

To verify this observe,

(i) That dZ'/dZ is never zero or infinite except at points on the real axis in the Z plane:

(ii) That if Z be real, and lie between two consecutive zeros or infinities of the function dZ'/dZ, say X_r and X_{r+1}, the argument of dZ'/dZ or dZ'/dX remains the same for all the values of X, so that the argument of dZ' remains the same, and all the points Z' which correspond to points on the real axis between X_r and X_{r+1} lie in one straight line in the Z' plane.

By combining (i) and (ii) it appears that the points on one side of the real axis in the Z plane correspond to points within a polygon in the Z' plane, and the points X_1, X_2, ... correspond to the corners.

We must further observe (iii) that if Z be very near to X_r all the other factors on the right of (B) may be considered constant except $(Z - X_r)^{\alpha_r/\pi - 1}$, so that in the neighbourhood of X_r the change of the argument of dZ'/dZ is the same as that of this factor—and, in passing through X_r in the positive direction, this change is an increase by $\pi - \alpha_r$, which is the same as the increase of argument of dZ' in going round a corner of the polygon where the internal angle is α_r.

We note that it is in general possible in a given problem to choose arbitrarily three of the points X_1, X_2, ... and then the rest will be determined.

2. The problems to which the theory is applicable are such as are concerned with the motion of fluid in space bounded partly by fixed plane rigid walls and partly by one or more free stream-lines along which the velocity is constant. These include the escape of a jet from a polygonal vessel, flow into

or out of straight tubes, flow past one or more plane laminas fixed in a finite stream whose boundaries are either free or fixed, or in an infinite stream. In all such cases there is a certain number of bounding stream-lines so that, with the notation of Prop. I., there will be a certain region in the z plane within which the motion takes place, bounded partly by given fixed straight lines and partly by unknown free stream-lines, and a certain corresponding region in the w plane bounded by parts of straight lines $\psi = \text{const.}$ If the motion and the form of the free stream-lines were known we should know the relation between w and z which effects a conformable representation of the region in the w plane upon the region in the z plane in which the motion takes place. Conversely if this relation can be found we shall know the motion and the form of the free stream-lines.

This relation can be found indirectly by the aid of the function Ω introduced in Prop. I. Along a plane rigid wall the direction of the velocity is given, and along a free stream-line the velocity is constant. Hence along a plane rigid wall the imaginary part of Ω is a given constant, and along the free stream-lines the real part of Ω is a given constant. There will exist a relation between Ω and z by which could be effected a conformable representation of a certain region in the Ω plane upon the region of the z plane within which the motion takes place. The region in the Ω plane is bounded by parts of straight lines parallel to the axis of Ω real, corresponding to the plane rigid walls, and parts of a straight line parallel to the axis of Ω imaginary, corresponding to the free stream-lines. The velocity vanishes at angles of the rigid boundary in the z plane, and at points where stream-lines divide; the corresponding points of the Ω plane lie at ∞ in the direction of Ω real, so that the general form of the boundary in the Ω plane is as in the figure,

where the thick lines correspond to rigid walls and the thin lines to free stream-lines, and there are as many thick lines as there are definite prescribed *directions* of motion. The intersections of thick and thin lines correspond to points where a free stream-line starts out from a rigid boundary.

The relation between Ω and z by which the representation of the Ω region upon the z region could be effected is unknown until the problem is solved, but, by applications of Schwarz's transformation given in Prop. II., the Ω region and the w region can each be represented upon the same half-plane in the plane of a new variable (u). In this way we can transform the Ω region into the w region and hence arises a relation $\Omega = f(w)$ or $\log(dz/dw) = f(w)$. This is a differential equation defining z as a function of w, and when it is solved we shall know the region of z which is represented upon the region of w. Part of the boundary of this z region is prescribed and will inevitably agree with that obtained by solution of the differential equation, the determination of the other part will give the form of the free stream-lines.

It is in general most convenient to take the constant velocity along the free stream-lines to be unity so that the corresponding value of the real part of Ω is zero, and to change the method of procedure sketched above by eliminating w and finding a differential equation between z and u. The region of the z plane within which the motion takes place will be that which by this relation is conformably represented upon the half-plane u.

In fact we have a given polygonal boundary (formed of parts of parallel lines) in the w plane, and this can be transformed into the real axis in the u plane by a relation of the form

$$\frac{dw}{du} = f_1(u).$$

And we have a given polygonal boundary in the Ω plane, and this can be transformed into the real axis in the u plane by a relation of the same form, say

$$\frac{d\Omega}{du} = f_2(u),$$

where the roots, poles and critical points of $f_2(u)$ are points $u = a, \ldots$ assumed to correspond to the corners of the polygon in the Ω plane. Integrating this equation we obtain

$$\frac{dz}{dw} = Ce^{\int f_2(u)\,du},$$

where C is a constant of integration. Hence we have the differential equation

$$\frac{dz}{du} = Cf_1(u)\,e^{\int f_2(u)\,du}.$$

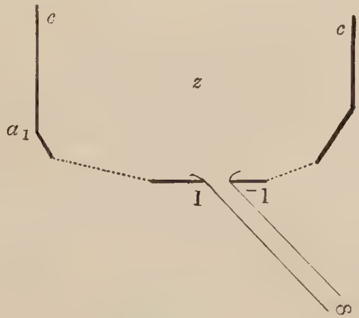
If we integrate this for the parts of the line u real that lie between the critical points of the function Ω, we shall find ex-

pressions for the lengths of the straight boundaries, which will determine the unknown constants a, and we shall find for z a complex expression when u has values corresponding to points on a free stream-line; *i.e.* the coordinates x and y of a point on a free stream-line are expressed as functions of a real parameter u.

All the problems solved in the first part of Mr Michell's paper can be treated in the manner here explained. I propose now to consider his first problem—that of the escape of a jet from a vessel, and then to give solutions applicable to other cases.

3. PROBLEM (i). *Escape of a jet from a vessel.*

Suppose we have a polygonal vessel of such shape that the fluid coming from ∞ must move parallel to the negative direction of the axis of y in the z plane, and the jet is formed by fluid escaping from a side parallel to the axis of x.



The boundary in the z plane will be as above, and as we are going to transform this boundary into a straight line u real in the plane of a new variable u it is convenient to denote points on the boundary by the values of u at the corresponding points. We shall assume then that the points $u=1$, $u=-1$ correspond to the edges of the hole, and $u=\infty$ to the point at ∞ in the z plane in the direction of the jet, and we shall take $u=c$, a_1, a_2, ... unknown constants for the points corresponding to the point at ∞ in the z plane from which the fluid comes and the corners of the polygonal boundary.

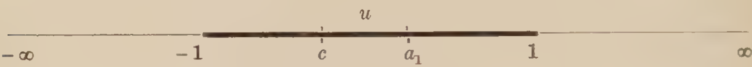
The boundary in the w plane consists of two infinite straight lines corresponding to the bounding stream-lines. If these be taken to be $\psi=0$, $\psi=\pi$, and the velocity along the free stream-lines be taken as unity, the ultimate breadth of the jet will be π.

The point $u = c$ will correspond to $\phi = -\infty$, and $u = \infty$ to $\phi = \infty$. The figure in the w plane is

where the lower line is the axis of real quantities $\psi = 0$, and the upper is $\psi = \pi$. The strip between these lines can be conformably represented upon a half-plane u by means of the transformation

$$\frac{dw}{du} = \frac{1}{u - c} \quad \text{.............................(1)},$$

and the boundary in the u plane is



4. *To find the boundary in the Ω plane.*

Let a_r be the value of u that corresponds to any angle α_r of the polygon. The stream-line $\psi = 0$ has for initial direction $\theta = -\frac{1}{2}\pi$. At the first angle a_1 the velocity vanishes, at ∞ it has a definite value. Thus $\log q^{-1}$ increases from a certain value to $+\infty$ as u moves from c to a_1, and θ remains constant and equal to $-\frac{1}{2}\pi$, and the corresponding part of the boundary in the Ω plane is therefore a straight line $\theta = -\frac{1}{2}\pi$ from a certain point for which the real part of Ω is positive to ∞.

As u increases from a_1 to a_2, $\log q^{-1}$ diminishes from ∞ to a certain minimum and then increases to ∞, θ remains positive and equal to $-\frac{1}{2}\pi + (\pi - \alpha_1)$. Thus we have the two sides of a second line starting from an unknown point (say $u = c_1$) and running to ∞ in the Ω plane in the direction of Ω real.

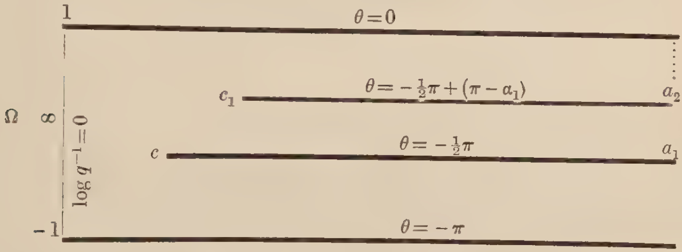
We shall obtain a line in the same way for each side of the vessel until we come to the side that contains the hole. On this line $\theta = 0$, and $\log q^{-1}$ diminishes from ∞ to zero, its value at the edge of the hole where the velocity is unity.

The next part of the boundary consists of the line $\log q^{-1} = 0$ drawn from the origin downwards (because θ becomes negative on the jet), to some value corresponding to $u = \infty$. This corresponds to the free part of the stream-line $\psi = 0$. Passing through this point the line can be continued downwards to $\theta = -\pi$ the value at the other edge of the hole corresponding to $u = -1$, this part of the line corresponds to the free part of the stream-line $\psi = \pi$.

Proceeding from this point we have to trace the line $\theta = -\pi$ to ∞, then both sides of the other lines $\theta = \text{const.}$ corresponding

to the further sides of the vessel, and finally the underside of the line $\theta = -\frac{1}{2}\pi$ from ∞ to the point corresponding to $u = c$.

Hence the boundary is



where the point marked 1 is the origin in the Ω plane and all points are marked by the corresponding values of u, and the figure is traced continuously by starting from any point, say c, going from c to a_1, from a_1 to c_1, from c_1 to a_2, and so on, keeping the region marked out on the left.

5. The Ω region can be conformably represented on the half-plane u by means of the relation

$$\frac{d\Omega}{du} = A \frac{u-c}{\sqrt{(u^2-1)}} \frac{(u-c_1)(u-c_2)\ldots}{(u-a_1)(u-a_2)\ldots} \quad \ldots\ldots\ldots\ldots(2),$$

where the number of the letters $a_1, a_2, \ldots$ exceeds that of the letters $c_1, c_2, \ldots$ by two, and A is a constant which can be determined; for the angles at 1 and -1 in the Ω plane are each $\frac{1}{2}\pi$, the angles at $a_1, a_2, \ldots$ are each 0, and the angles at $c, c_1, \ldots$ are each 2π.

Now the function

$$\frac{(u-c)(u-c_1)(u-c_2)\ldots}{(u-a_1)(u-a_2)\ldots}$$

can be put into partial fractions in the form

$$\frac{A_1}{u-a_1} + \frac{A_2}{u-a_2} + \ldots$$

where $A_1, A_2, \ldots$ depend only on the c's and a's, and thus we may write (2) in the form

$$\frac{d\Omega}{du} = -Ai \sum_n \left[\frac{A_n}{(u-a_n)} \frac{1}{\sqrt{(1-u^2)}}\right] \quad \ldots\ldots\ldots\ldots(3),$$

and the integral of this is

$$\Omega = -Ai \log\left\{B \prod_n \left[\frac{(1-a_n u) + \sqrt{(1-a_n^2)}\ \sqrt{(1-u^2)}}{u-a_n}\right]^{A_n/\sqrt{(1-a_n^2)}}\right\} \quad \ldots\ldots\ldots\ldots(4),$$

where B is a constant which can be determined.

Remembering that $\Omega = \log (dz/dw)$ and using (1) we find

$$\frac{dz}{du} = \frac{B}{u-c} \prod_n \left[\frac{(1-a_n u) + \sqrt{(1-a_n^2)}\sqrt{(1-u^2)}}{u-a_n}\right]^{-Ai\,.\,A_n/\sqrt{(1-a_n^2)}} \quad \text{...............(5).}$$

To determine the constants observe first that in passing the point a_n the argument of u increases by $\pi - \alpha_n$, so that the index $-Ai\,.\,A_n/\sqrt{(1-a_n^2)}$ must be $1-\alpha_n/\pi$ and the sum of these indices is unity. Also if $u=1$ each factor in the product $=1$ and $\Omega = \log B$ is zero so that $B=1$.

The differential equation for z thus becomes

$$\frac{dz}{du} = \frac{1}{u-c} \prod_n \left[\frac{(1-a_n u) + \sqrt{(1-a_n^2)}\sqrt{(1-u^2)}}{u-a_n}\right]^{1-\alpha_n/\pi} \quad \text{......(6),}$$

as in Mr Michell's paper, p. 401.

We have also a series of relations of the form

$$1 - \frac{\alpha_n}{\pi} = \frac{-Ai\,.\,A_n}{\sqrt{(1-a_n^2)}} = -\frac{Ai}{\sqrt{(1-a_n^2)}} \frac{(a_n-c)(a_n-c_1)(a_n-c_2)\dots}{(a_n-a_1)(a_n-a_2)\dots} \quad \text{...............(7).}$$

There are as many of these as there are a's, that is, as many as the number of the c's $+1$ and these would suffice to determine the c's and A in terms of the a's.

In the particular problem worked out in detail by Mr Michell, viz. that of two rectangular corners a and b for which

$$1 > b > c > a > -1,$$

the above relations become

$$-\frac{Ai}{\sqrt{(1-b^2)}} \frac{b-c}{b-a} = -\frac{Ai}{\sqrt{(1-a^2)}} \frac{a-c}{a-b} = \frac{1}{2},$$

giving a relation

$$\frac{a-c}{\sqrt{(1-a^2)}} = \frac{c-b}{\sqrt{(1-b^2)}} \quad \text{.....................(8),}$$

which determines c in terms of a and b, viz.

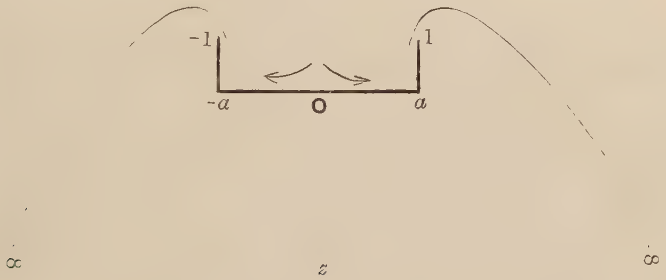
$$c = \frac{a\sqrt{(1-b^2)} + b\sqrt{(1-a^2)}}{\sqrt{(1-a^2)} + \sqrt{(1-b^2)}} \quad \text{..................(9).}$$

This relation must hold among the constants of Mr Michell's problem. In his paper c is only determined in the case of symmetry for which it is obviously zero.

I do not propose to proceed further with this problem at present. (See Art. 13 below.)

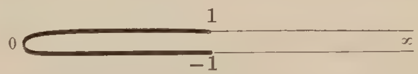
6. PROBLEM (ii). *Liquid flowing against a disc with an elevated rim.*

Suppose that in an infinite mass of fluid moving from infinity in the negative direction of the axis of y there is a vessel bounded by three sides of a rectangle the missing side being parallel to the axis of x. The fluid will enter the vessel—one stream-line will divide at the middle point of the base, and there will be two free stream-lines starting out from the edges of the vessel, behind which there will be dead water.



The boundary in the z plane will be as in the figure, and we shall suppose the points $u = \pm 1$ to correspond to the edges of the vessel, $u = \pm a$ to the corners of the vessel, and $u = 0$ to the point where the stream-line divides.

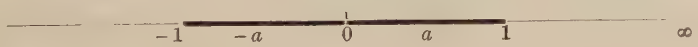
The boundary in the w plane will consist of the two sides of the line $\psi = 0$ from the origin to $+\infty$, thus



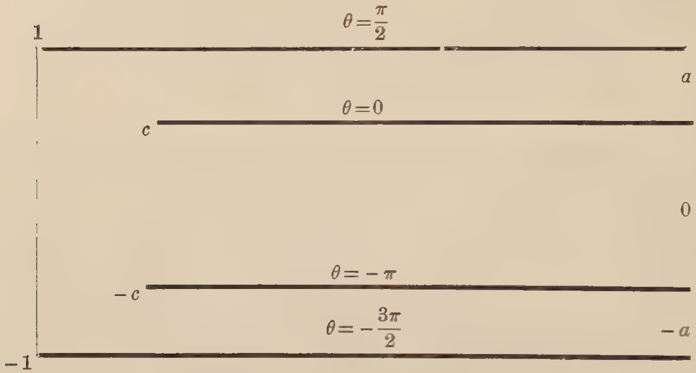
This can be transformed into the real axis in the u plane by taking

$$\frac{dw}{du} = \frac{1}{2}u \quad \text{.........................(10)},$$

and then the u boundary is



The Ω boundary is



where c and $-c$ are the values of u at the points in the base of the vessel where the velocity is a maximum.

7. This Ω region can be conformably represented upon the half-plane u by means of the relation

$$\frac{d\Omega}{du} = A\,\frac{u^2 - c^2}{\sqrt{(u^2-1)}\,.\,u\,(u^2 - a^2)} \quad\text{..............(11).}$$

Writing this in partial fractions

$$\frac{d\Omega}{du} = \frac{-Ai}{\sqrt{(1-u^2)}}\left[\frac{c^2}{a^2}\frac{1}{u} + \frac{a^2-c^2}{2a^2}\frac{1}{u-a} + \frac{a^2-c^2}{2a^2}\frac{1}{u+a}\right],$$

and integrating we have

$$\Omega = \log\left\{B\left[\frac{1+\sqrt{(1-u^2)}}{u}\right]^{\frac{-Aic^2}{a^2}}\right.$$

$$\left.\left[\frac{1-au+\sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u-a}\,\frac{1+au+\sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u+a}\right]^{\frac{-Ai(a^2-c^2)}{2a^2\sqrt{(1-a^2)}}}\right\}$$

so that

$$\frac{dz}{du} = \frac{1}{2}u\,.\,B\,.\left[\frac{1+\sqrt{(1-u^2)}}{u}\right]^{-\frac{Aic^2}{a^2}}$$

$$\left[\frac{1-au+\sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u-a}\,\frac{1+au+\sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u+a}\right]^{\frac{-Ai(a^2-c^2)}{2a^2\sqrt{(1-a^2)}}}$$

......(12).

Now as in the preceding, since the corners of the z polygon that correspond to 1 and -1 are right angles

$$\frac{-Ai(a^2-c^2)}{2a^2\sqrt{(1-a^2)}} = \frac{1}{2},$$

and since the z boundary has no singularity at $u=0$

$$-\frac{Aic^2}{a^2}=1,$$

whence $$\frac{a^2-c^2}{c^2}=\sqrt{(1-a^2)} \text{ or } c^2=\frac{a^2}{1+\sqrt{(1-a^2)}} \quad\text{.........(13).}$$

Also $\Omega=+i\frac{\pi}{2}$ when $u=1$ so that $B=i$, and the relation between z and u is

$$\frac{dz}{du}=\frac{i}{2}\{1+\sqrt{(1-u^2)}\}$$
$$\left[\frac{1-au+\sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u-a}\,\frac{1+au+\sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u+a}\right]^{\frac{1}{2}}.$$

Now remembering the identity,

$$\frac{\sqrt{(1+a)}\sqrt{(1+u)}+\sqrt{(1-a)}\sqrt{(1-u)}}{\sqrt{2}}=\{1+au+\sqrt{(1-a^2)}\sqrt{(1-u^2)}\}^{\frac{1}{2}},$$

we have

$$\frac{dz}{du}=\frac{i}{2}[1+\sqrt{(1-u^2)}]\,\frac{\sqrt{(1-a^2)}+\sqrt{(1-u^2)}}{\sqrt{(u^2-a^2)}} \quad\text{......(14).}$$

This corresponds to the case of liquid flowing directly against a disc with an elevated rim, the rim being the line in the z plane that joins the points corresponding to $u=a$ and $u=1$. Observe by way of verification that if we make $a=1$ there will be no rim and we get the right transformation for fluid flowing against a plane disc, viz.:

$$\frac{dz}{du}=\frac{1}{2}[1+\sqrt{(1-u^2)}] \quad\text{..................(15).}$$

8. Suppose now that a is very nearly 1 and take

$$1-a^2=k^2,\ a^2=1-k^2=k'^2 \quad\text{..................(16),}$$

where k is small. Then equation (14) may be written

$$\frac{dz}{du}=\frac{i}{2}\left[\frac{\sqrt{(1-u^2)}}{\sqrt{(u^2-a^2)}}\{1+\sqrt{(1-a^2)}\}+\frac{\sqrt{(1-a^2)}}{\sqrt{(u^2-a^2)}}+\frac{1-u^2}{\sqrt{(u^2-a^2)}}\right],$$

and the height of the rim is

$$\frac{1}{2}\int_a^1\left[\frac{\sqrt{(1-u^2)}}{\sqrt{(u^2-a^2)}}\{1+\sqrt{(1-a^2)}\}+\frac{\sqrt{(1-a^2)}}{\sqrt{(u^2-a^2)}}+\frac{1-u^2}{\sqrt{(u^2-a^2)}}\right]du \quad\text{......(17).}$$

Now $$\int_a^1 \frac{\sqrt{(1-u^2)}}{\sqrt{(u^2-a^2)}}\,du = \int_a^1 \frac{(1-u^2)\,du}{\sqrt{(1-u^2)}\sqrt{(u^2-a^2)}},$$

putting $u = a\sec\phi$, the latter integral becomes

$$\int_0^{\cos^{-1}a} \frac{(1-a^2\sec^2\phi)\,a\sec\phi\tan\phi\,d\phi}{a\sqrt{(1-a^2\sec^2\phi)}\tan\phi} = \int_0^{\cos^{-1}a} \frac{(1-a^2\sec^2\phi)\,d\phi}{\sqrt{(1-a^2-\sin^2\phi)}},$$

putting $\sin\phi = \sqrt{(1-a^2)}\sin\psi = k\sin\psi$, we have

$$\int_0^{\frac{\pi}{2}} \frac{k^2\cos^2\psi\,d\psi}{[1-k^2\sin^2\psi]^{\frac{3}{2}}},$$

or $k^2\dfrac{\pi}{4}$ in the limit when k is small.

Again, $$\int_a^1 \frac{du}{\sqrt{(u^2-a^2)}} = \int_0^{\cos^{-1}a} \sec\phi\,d\phi$$
$$= k \text{ in the limit}$$

and $$\int_a^1 \frac{(1-u^2)}{\sqrt{(u^2-a^2)}}\,du = \int_0^{\cos^{-1}a} (1-a^2\sec^2\phi)\sec\phi\,d\phi$$

$$= \int_0^{\cos^{-1}a} \sec\phi\,d\phi - \frac{a^2}{2}\left\{[\sec\phi\tan\phi]_0^{\cos^{-1}a} + \int_0^{\cos^{-1}a} \sec\phi\,d\phi\right\}$$

$$= \frac{1+k^2}{2}\log\sqrt{\frac{1+k}{1-k}} - \frac{1}{2}k$$

$$= 0 \text{ as far as } k^2.$$

Thus neglecting k^3 the height of the rim is

$$\frac{1}{2}k^2\left(1+\frac{\pi}{4}\right).$$

To find the breadth of the disc we must write (14) in the form

$$\frac{dz}{du} = \frac{1}{2}[1+\sqrt{(1-u^2)}]\frac{k+\sqrt{(1-u^2)}}{\sqrt{(1-u^2-k^2)}},$$

and integrate from $u = 0$ to $u = a$ and double the result. We may reject k altogether, and thus find the breadth

$$a + \tfrac{1}{2}[\sin^{-1}a + a\sqrt{(1-a^2)}],$$

or in the limit when a is very nearly 1 we may take the breadth equal to

$$\frac{4+\pi}{4} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

thus the ratio height of rim to breadth of disc $= \frac{1}{2}k^2 = \epsilon$ say.

When $u > 1$, we must write (14) in the form

$$\frac{dz}{du} = \frac{i}{2}[1 - i\sqrt{(u^2-1)}]\frac{k - i\sqrt{(u^2-1)}}{\sqrt{(u^2-1+k^2)}}$$

$$= \frac{1}{2}(1+k)\frac{\sqrt{(u^2-1)}}{\sqrt{(u^2-1+k^2)}} + \frac{i}{2}\left[\frac{k}{\sqrt{(u^2-1+k^2)}} - \frac{u^2-1}{\sqrt{(u^2-1+k^2)}}\right].$$

For the imaginary part of z taking $\cosh\theta = u/k'$ we have

$$\frac{i}{2}\left[k\theta - \left\{\frac{k'^2}{2}(\theta + \sinh\theta\cosh\theta) - \theta\right\}\right] + \text{const.},$$

or writing $z = x + \iota y$

$$y = \frac{1}{2}\left[(1+k)\,\theta - \frac{1-k^2}{2}(\theta + \sinh\theta\cosh\theta)\right] + \text{const.}$$

The greatest value is given at once by $\dfrac{dy}{du} = 0$, so that

$$u^2 - 1 = k,$$

or $$\cosh^2\theta = \frac{k+1}{1-k^2} = \frac{1}{1-k},$$

giving $$\sinh\theta = \sqrt{k}\,(1 + \tfrac{1}{2}k),$$

$$\theta = \sqrt{k}\,(1 + \tfrac{1}{3}k).$$

The value corresponding to $u = 1$ is given by

$$\cosh^2\theta = \frac{1}{1-k^2},$$

giving $$\sinh\theta = k\,(1 + \tfrac{1}{2}k^2),$$

$$\theta = k\,(1 + \tfrac{1}{3}k^2).$$

Hence the height above the rim through which the free stream-line rises before turning back is

$$\tfrac{1}{2}(1+k)\sqrt{k}\,(1+\tfrac{1}{3}k) - \tfrac{1}{4}\sqrt{k}\,[(1+\tfrac{1}{3}k) + (1+\tfrac{1}{2}k)(1+k)]$$
$$- \tfrac{1}{2}(1+k)\,k\,(1+\tfrac{1}{3}k^2) + \tfrac{1}{4}k\,[(1+\tfrac{1}{3}k^2) + (1+\tfrac{1}{2}k^2)(1+k^2)],$$

and this is ultimately

$$\tfrac{5}{24}k^{\frac{3}{2}},$$

rejecting higher powers of k.

Hence if ϵ be the ratio height of rim to breadth of disc the greatest height above the rim to which the free stream-line rises before turning back is

$$\tfrac{5}{24}(2\epsilon)^{\frac{3}{4}} \text{ of the breadth} \qquad\text{.....................(19)},$$

or about $$\tfrac{1}{3}\epsilon^{\frac{3}{4}} \text{ of the breadth},$$

e.g. if the rim be $\frac{1}{16}$th of the breadth the stream-line rises through about $\frac{1}{24}$th of the breadth. Except quite close to the disc where u^2-1 is small we may reject k altogether, and the form of the free stream-line at a distance from the disc is the same as if there were no rim.

To find the pressure on the disc we have to find

$$\rho\int_0^a\left[1-\left(\frac{dw}{dz}\right)^2\right]\frac{dz}{du}\,du,$$

which is

$$\tfrac{1}{2}\rho\int_0^a\left\{1-\frac{u^2(1-u^2-k^2)}{[1+\sqrt{(1-u^2)}]^2\,[k+\sqrt{(1-u^2)}]^2}\right\}\frac{k+\sqrt{(1-u^2)}}{\sqrt{(1-u^2-k^2)}}\,[1+\sqrt{(1-u^2)}]\,du,$$

$$\text{or}\quad \tfrac{1}{2}\rho\int_0^{\sin^{-1}a}\frac{(1+\cos\theta)^2(k+\cos\theta)^2-\sin^2\theta\cos^2\theta}{(1+\cos\theta)(k+\cos\theta)}\,d\theta,\quad\text{rejecting } k^2,$$

$$=\tfrac{1}{2}\rho\int_0^{\sin^{-1}a}[(1+\cos\theta)(k+\cos\theta)-(1-\cos\theta)\cos\theta\,(1-k\sec\theta)]\,d\theta$$

$$=\tfrac{1}{2}\rho\int_0^{\sin^{-1}a}(2\cos^2\theta+2k)\,d\theta$$

$$=\tfrac{1}{2}\rho\,[\sin^{-1}a+a\sqrt{(1-a^2)}+2k\sin^{-1}a]$$

$$=\rho\left[\frac{\pi}{4}+\frac{k}{2}+\frac{k\pi}{2}\right]\ \text{to the first order in } k.$$

The breadth of the disc, rejecting k^2, is

$$a+\tfrac{1}{2}[\sin^{-1}a+a\sqrt{(1-a^2)}]+k[a+\sin^{-1}a],$$

$$\text{or}\qquad 1+\frac{\pi}{4}+\frac{k}{2}+k+\frac{\pi}{2}k\ \text{ to the first order in } k.$$

Thus the mean pressure, when the velocity on the free stream-lines is V, is

$$\rho V^2\left[\frac{\pi}{4+\pi}+k\,\frac{8+4\pi+2\pi^2}{(4+\pi)^2}\right]$$

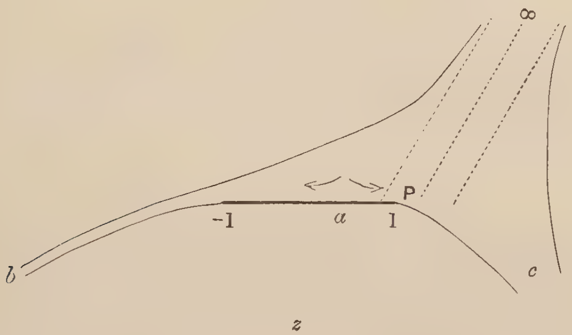
$$\text{or}\quad \frac{\pi\rho V^2}{4+\pi}\left[1+\frac{8+4\pi+2\pi^2}{(4+\pi)^2}\sqrt{(2\epsilon)}\right]\ \ldots(20).$$

Comparing this with the case where there is no rim we see that the mean pressure is increased by about $\frac{6}{5}\sqrt{\epsilon}$ of itself, ϵ being the ratio height of rim to breadth of disc.

9. PROBLEM (iii). *Oblique impact of a jet upon a finite lamina.*

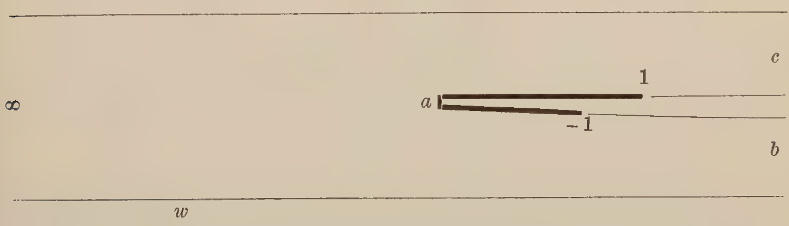
This problem includes Mr Michell's of the impact of a jet against an infinite plane, and also the problem of the impact of an infinite stream against a finite lamina worked out by Kirchhoff and Lord Rayleigh.

The z-region is bounded by two free stream-lines, by the lamina, and by the continuations of the stream-line that divides on the lamina.



We shall suppose that $u = \pm 1$ correspond to the ends of the lamina, and $u = a$ to the point where the stream-line divides, $u = b$ and $u = c$ to the points to which the two streams go.

The w-region will be bounded by the two infinite lines $\psi = 0$ and $\psi = \pi$ say, making the breadth of the original jet π, and by the two sides of a line $\psi = \beta$ say, where $\pi > \beta > 0$; thus

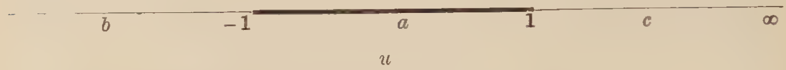


and this region can be conformably represented upon the half-plane u by means of the relation

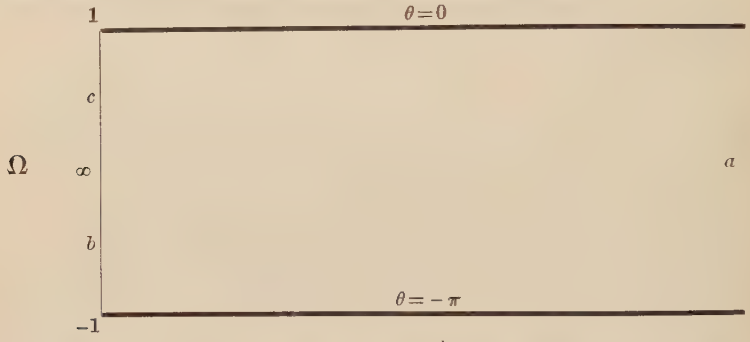
$$\frac{dw}{du} = -\frac{u-a}{(u-b)(u-c)} \qquad\qquad (21),$$

and then $$\psi = \frac{a-b}{c-b}\pi$$

will be the stream-line that divides, and the boundary in the u plane is



The boundary in the Ω plane is easily seen to be



This Ω-region can be conformably represented upon the half-plane u by means of the relation

$$\frac{d\Omega}{du} = A\frac{1}{\sqrt{(u^2-1)}}\frac{1}{u-a} \quad\text{.......................}(21),$$

whence $\quad \Omega = \log\left[\dfrac{1-au+\sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u-a}\right]^{-\frac{Ai}{\sqrt{(1-a^2)}}},$

so that $\quad \dfrac{dz}{du} = -\dfrac{1-au+\sqrt{(1-a^2)}\sqrt{(1-u^2)}}{(u-b)(u-c)} \quad$ ....(22),

in which the index $-\dfrac{Ai}{\sqrt{(1-a^2)}}$ has been determined to be unity by considering that there is no singularity in the z boundary at the point corresponding to $u = a$.

10. We may now suppose that the lamina is part of the axis of x. If we take u to lie between -1 and 1 and integrate (22) between these limits we shall get an expression for the breadth of the lamina.

Writing the equation (22) in the form

$$\frac{dz}{du} = -\frac{ab-1}{c-b}\frac{1}{u-b} - \frac{ac-1}{c-b}\frac{1}{c-u}$$

$$+\frac{\sqrt{(1-a^2)}}{\sqrt{(1-u^2)}} - \frac{\sqrt{(1-a^2)}}{c-b}\left[\frac{b^2-1}{(u-b)\sqrt{(1-u^2)}} + \frac{c^2-1}{(c-u)\sqrt{(1-u^2)}}\right]$$

we find for the breadth of the lamina

$$\frac{ac-1}{c-b}\log\frac{c-1}{c+1}-\frac{ab-1}{c-b}\log\frac{1-b}{-1-b}$$
$$+\sqrt{(1-a^2)}\,\pi-\frac{\sqrt{(1-a^2)}}{c-b}[\sqrt{(b^2-1)}+\sqrt{(c^2-1)}]\,\pi\ldots(23).$$

When u is >1 we must write (22) in the form

$$\frac{dz}{du}=\frac{au-1+i\sqrt{(1-a^2)}\sqrt{(u^2-1)}}{(u-b)(u-c)}\quad\ldots\ldots\ldots(24),$$

where $a=\cos\alpha$ and $\sqrt{(1-a^2)}=\sin\alpha$, α being the angle the impinging jet makes with the lamina. The sign of the imaginary term is determined by considering that when $u=\infty$ the argument of dz is α.

When u lies between 1 and c we have

$$\frac{dz}{du}=-\frac{ab-1}{c-b}\frac{1}{u-b}-\frac{ac-1}{c-b}\frac{1}{c-u}+i\frac{\sqrt{(1-a^2)}}{\sqrt{(u^2-1)}}$$
$$-i\frac{\sqrt{(1-a^2)}}{(c-b)\sqrt{(u^2-1)}}\left[\frac{b^2-1}{u-b}+\frac{c^2-1}{c-u}\right],$$

and the argument of dz in the neighbourhood of $u=c$ is the same as that of

$$-\frac{ac-1}{c-b}-i\frac{\sqrt{(1-a^2)}}{\sqrt{(c^2-1)}(c-b)},$$

or of
$$-(ac-1)-i\sqrt{(1-a^2)}/\sqrt{(c^2-1)}.$$

Thus the ultimate direction of this part of the stream is θ, where

$$R\cos\theta=1-ac,$$
$$R\sin\theta=-\sqrt{(1-a^2)}/\sqrt{(c^2-1)},$$

giving
$$R=c-a,$$

and
$$\tan\theta=-\frac{\sqrt{(1-a^2)}}{(1-ac)\sqrt{(c^2-1)}}\quad\ldots\ldots\ldots\ldots\ldots(25).$$

In like manner the ultimate direction of the other part may be determined.

11. The form of the free stream-line from 1 to c would be found by integrating (24). I shall suppose $u=+1$ to correspond

to $z=0$, and then the co-ordinates of a point on the free stream-line are

$$\left.\begin{aligned} x &= \frac{ac-1}{c-b}\log\frac{c-u}{c-1} - \frac{ab-1}{c-b}\log\frac{u-b}{1-b} \\ y &= \sqrt{(1-a^2)}\cosh^{-1}u - \frac{\sqrt{(1-a^2)}}{c-b}\left[\sqrt{(b^2-1)}\sinh^{-1}\left\{\frac{\sqrt{(b^2-1)}\sqrt{(u^2-1)}}{u-b}\right\}\right. \\ &\qquad \left. + \sqrt{(c^2-1)}\sinh^{-1}\frac{\sqrt{(c^2-1)}\sqrt{(u^2-1)}}{c-u}\right] \end{aligned}\right\} \quad (26),$$

in which u is a real parameter lying between 1 and c.

To find the free stream-lines that bound the jet, we have to integrate (24) up to values of u lying beyond $u=c$. We choose the path of integration to start from $u=1$ and proceed along u real very nearly to $u=c$, then over a little semicircle whose centre is $u=c$ and then again along u real; we thus find

$$\begin{aligned} z &= -\frac{ab-1}{c-b}\log\frac{u-b}{1-b} + \frac{ac-1}{c-b}\log\frac{u-c}{c-1} \\ &+ i\sqrt{(1-a^2)}\cosh^{-1}u - i\frac{\sqrt{(1-a^2)}}{c-b}\left[\sqrt{(b^2-1)}\sinh^{-1}\left\{\frac{\sqrt{(b^2-1)}\sqrt{(u^2-1)}}{u-b}\right\}\right. \\ &\left. - \sqrt{(c^2-1)}\sinh^{-1}\left\{\frac{\sqrt{(c^2-1)}\sqrt{(u^2-1)}}{u-c}\right\}\right] \\ &- i\pi\left[\frac{ac-1}{c-b} + i\frac{\sqrt{(1-a^2)}\sqrt{(c^2-1)}}{c-b}\right], \end{aligned}$$

where the last line is the part contributed to the integral by the small semicircle. Hence on this stream-line

$$\left.\begin{aligned} x &= \frac{ac-1}{c-b}\log\frac{u-c}{c-1} - \frac{ab-1}{c-b}\log\frac{u-b}{1-b} + \frac{\pi\sqrt{(1-a^2)}\sqrt{(c^2-1)}}{c-b} \\ y &= \sqrt{(1-a^2)}\cosh^{-1}u - \frac{\sqrt{(1-a^2)}}{c-b}\left[\sqrt{(b^2-1)}\sinh^{-1}\left\{\frac{\sqrt{(b^2-1)}\sqrt{(u^2-1)}}{u-b}\right\}\right. \\ &\qquad \left. - \sqrt{(c^2-1)}\sinh^{-1}\left\{\frac{\sqrt{(c^2-1)}\sqrt{(u^2-1)}}{u-c}\right\}\right] - \pi\frac{ac-1}{c-b} \end{aligned}\right\} \quad (27).$$

This stream-line will have an asymptote $x = y\cot\alpha + x_0$ which is to be determined by making u infinite and

$$a = \cos\alpha, \quad \sqrt{(1-a^2)} = \sin\alpha.$$

When u is very great, we can expand x and y in the forms

$$x = \qquad a \log u + A_0 + \frac{A_1}{u} + \ldots$$

$$y = \sqrt{(1-a^2)} \log u + B_0 + \frac{B_1}{u} + \ldots$$

and, to find the asymptote, we must find A_0 and B_0. We get

$$x = \qquad a \log u + \left[\frac{ab-1}{c-b} \log(1-b) - \frac{ac-1}{c-b} \log(c-1) + \frac{\pi\sqrt{(1-a^2)}\sqrt{(c^2-1)}}{c-b}\right] + \text{terms in } \frac{1}{u},$$

$$y = \sqrt{(1-a^2)} \log u + \left[\sqrt{(1-a^2)} \log 2 - \frac{\sqrt{(1-a^2)}}{c-b}\{\sqrt{(b^2-1)} \cosh^{-1}(-b) - \sqrt{(c^2-1)} \cosh^{-1} c\} - \pi \frac{ac-1}{c-b}\right]$$
$$+ \text{terms in } \frac{1}{u},$$

so that

$$x_0 = \frac{ab-1}{c-b} \log(1-b) - \frac{ac-1}{c-b} \log(c-1) + \frac{\pi\sqrt{(1-a^2)}\sqrt{(c^2-1)}}{c-b}$$
$$- a \log 2 - \frac{a}{c-b}\{\sqrt{(b^2-1)} \cosh^{-1}(-b) - \sqrt{(c^2-1)} \cosh^{-1} c\}$$
$$+ \frac{a\pi}{\sqrt{(1-a^2)}} \frac{ac-1}{c-b} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28),$$

and $x_0 - \frac{1}{2}\pi \operatorname{cosec} \alpha$ is the distance to the right of the right-hand end of the lamina of the point where the initial middle line of the jet strikes the plane of the lamina. This is the point marked P in the z figure.

There are now sufficient relations to determine the unknown constants a, b, c. A jet of given breadth coming from ∞ in a given direction strikes the lamina obliquely in such a way that the middle line of the jet passes through a certain point in the lamina. We choose the unit of length so that the given breadth of the jet may be π. Then the constant a is determined by making $a = \cos \alpha$, $\sqrt{(1-a^2)} = \sin \alpha$, where α is the angle between the direction of the impinging jet and the lamina; and there are two relations to determine the constants b and c, viz., equation (23) gives the breadth of the lamina in terms of b and c, and equation (28) enables us to identify a point P whose distance from the right of (1) is given with a point whose distance from the right of (1) is a given function of b and c.

The particular cases investigated by Kirchhoff and Lord Rayleigh include all that have any practical interest at all comparable with the analytical difficulties.

12. Problem (iv). *Plane obstacle situated in canal of finite width.*

A stream flowing between two fixed parallel planes impinges upon a lamina fixed across the stream at a given angle α.

The z-region is bounded by the two fixed planes, the lamina, and the continuations beyond its edges of the stream-lines that

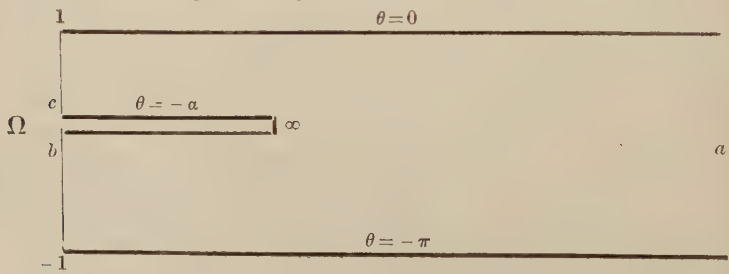
divide upon the lamina. We shall suppose the point at ∞ from which the stream comes to correspond to $u=\infty$; the points to which its two parts go, to $u=b, c$; the point where the stream-line divides, to $u=a$; and the edges of the lamina to $u=\pm 1$. Then we must have

$$c > 1 > a > -1 > b.$$

The w and u regions will be precisely the same as in the last problem, and the relation between them will be

$$\frac{dw}{du} = -\frac{u-a}{(u-b)(u-c)} \quad \text{.....................(29).}$$

The Ω boundary is easily seen to be



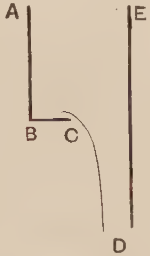
where the point marked ∞ corresponds to the point from which the stream comes and the point marked 1 is the origin in the Ω plane.

Now this polygon has right angles at $1, -1, c, b$, an angle 0 at a and an angle 2π at ∞ and it can be conformably represented upon the half-plane u by means of the relation

$$\frac{d\Omega}{du} = \frac{A}{\sqrt{\{(u^2-1)(u-b)(u-c)\}}\,(u-a)} \quad \text{.........(30)}.$$

In the general case the integration will require elliptic functions and I do not propose to proceed with it.

13. In the case of symmetry it is clear that there is no flow across the line perpendicular to the lamina at its middle point so that this line may be treated as a real boundary and the z-region will be



and the image of this in the line AB. It is clear that the motion takes place as if the boundary were $ABCDE$. If now we reflect this in the line DE we get the same figure as in the escape of a jet from a rectangular vessel by an orifice in the middle of the base. This is a particular case of our first problem and the form of the jet has been worked out by Mr Michell.

The relations between z, w, and u, are

$$\left.\begin{aligned} \frac{dw}{du} &= \frac{1}{u} \\ \frac{dz}{du} &= \frac{1}{u}\,\frac{\sqrt{(1-a^2)}+\sqrt{(1-u^2)}}{\sqrt{(u^2-a^2)}} \end{aligned}\right\} \quad \text{...............(31)},$$

and we propose to find the pressure on the side of the vessel in which the aperture is. If we take the part to the left of the aperture we shall have for the difference of pressure on the two sides of this plane

$$\tfrac{1}{2}\rho \int_a^1 \left[1 - \left(\frac{dw}{dz}\right)^2\right] \frac{dz}{du}\, du \quad \text{...................(32)}$$

where ρ is the density of the fluid, or

$$\tfrac{1}{2}\rho\int_a^1\left[1-\frac{u^2-a^2}{2-a^2-u^2+2\sqrt{(1-a^2)}\sqrt{(1-u^2)}}\right].\frac{\sqrt{(1-a^2)}+\sqrt{(1-u^2)}}{\sqrt{u^2-a^2}}.\frac{du}{u},$$

or
$$\rho\int_a^1\frac{\{\sqrt{(1-u^2)}+\sqrt{(1-a^2)}\}\sqrt{(1-u^2)}}{\{\sqrt{(1-a^2)}+\sqrt{(1-u^2)}\}\sqrt{(u^2-a^2)}}.\frac{du}{u},$$

or
$$\rho\int_a^1\frac{\sqrt{(1-u^2)}}{\sqrt{(u^2-a^2)}}\frac{du}{u}.$$

The value of the integral is

$$\rho.\frac{\pi}{2}\left(\frac{1}{a}-1\right).$$

Returning now to the problem of the lamina we see that the pressure on it is

$$\rho\pi(1-a)/a \quad\text{.........................(33)},$$

and in the same case the velocity of the stream at infinity in the direction from which it comes is

$$a/\{1+\sqrt{(1-a^2)}\} \quad\text{.........................(34)},$$

the breadth of the stream is

$$d=\pi\{1+\sqrt{(1-a^2)}\}/a \quad\text{..................(35)},$$

and the breadth of the lamina is

$$l=\pi\frac{1-a}{a}+\frac{\sqrt{(1-a^2)}}{a}(\pi-2\sin^{-1}a) \quad\text{............(36)}.$$

Now let the stream flow from ∞ with a given velocity V in a canal of breadth d, and impinge symmetrically and directly on a pier of breadth l. Then if a quantity a be determined from the equation

$$\frac{l}{d}=\frac{1-a+\sqrt{(1-a^2)}\left(1-\frac{2}{\pi}\sin^{-1}a\right)}{\sqrt{(1-a^2)}},$$

the pressure on the pier will be

$$\rho\pi V^2 l\frac{(1-a)\{1+\sqrt{(1-a^2)}\}^2}{a^2\{\pi(1-a)+\sqrt{(1-a^2)}(\pi-2\sin^{-1}a)\}}.$$

By writing $a=\cos\alpha$ we find the convenient form

$$\frac{l}{d}=\tan\tfrac{1}{2}\alpha+\frac{2\alpha}{\pi} \quad\text{.........................(37)},$$

and the mean pressure is

$$\frac{\rho\pi V^2(\sec^2\alpha-\sec\alpha)(1+\sin\alpha)^2}{\pi(1-\cos\alpha)+2\alpha\sin\alpha}\quad\ldots\ldots\ldots\ldots\ldots(38),$$

where α is determined by (37).

14. When the sides of the canal are distant, l is small compared with d, and if we take $l/d=\epsilon$ we shall have

$$\epsilon=\frac{2\alpha}{\pi}+\frac{1}{2}\alpha+\frac{1}{24}\alpha^3+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(39),$$

so that

$$\alpha=\frac{2\pi}{\pi+4}\epsilon-\frac{1}{24}\left(\frac{2\pi}{\pi+4}\right)^4\epsilon^3 \text{ nearly}\ldots\ldots\ldots\ldots(40).$$

As a first approximation taking α small, we have for the mean pressure

$$\frac{\rho\pi V^2(\frac{1}{2}\alpha^2)}{\pi\frac{\alpha^2}{2}+2\alpha^2}=\frac{\rho\pi V^2}{4+\pi}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(41),$$

the same as for a lamina held in an infinite stream. Going to a second approximation we have merely to retain the term 2α in the expansion of $(1+\sin\alpha)^2$ and this gives for the mean pressure

$$\frac{\rho\pi V^2}{4+\pi}\left[1+\frac{4\pi}{4+\pi}\epsilon\right]\ldots\ldots\ldots\ldots\ldots\ldots\ldots(42),$$

so that the effect of the sides of the canal is to increase the mean pressure by

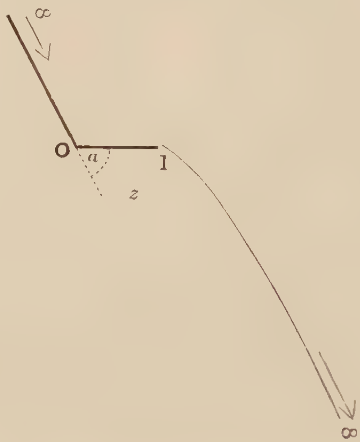
$$\frac{4\rho\pi^2V^2}{(4+\pi)^2}\epsilon\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(43),$$

and the fraction of itself by which the mean pressure is increased is about $\frac{7}{4}\epsilon$, in which it is to be remembered that ϵ is the ratio, breadth of pier to breadth of canal.

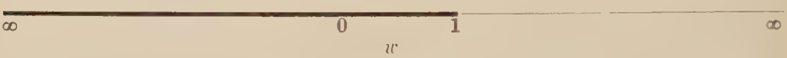
15. Problem (v). *Stream flowing past an obliquely projecting pier.*

Suppose the z boundary consists of parts of two straight lines one of them infinite in one direction and terminated at the point marked O, and the other finite and inclined to the first at a given angle $\pi-\alpha$, and that the fluid is on the side of the boundary

within the angle $\pi - \alpha$ and comes from ∞ in the direction indicated. The figure in the z plane is

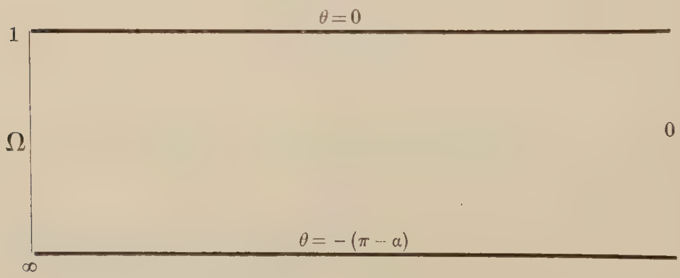


There being but one bounding stream-line, the figure in the w plane is



so that there is no necessity to transform to a new u plane. We take then the point $w = 0$ to be the corner, $w = 1$ the extremity of the broken line and $w = \infty$ the point to which the stream goes.

The figure in the Ω plane is



and this can be conformably represented on the half-plane w by means of the relation

$$\frac{d\Omega}{dw} = \frac{A}{\sqrt{(w-1)} \cdot w} \quad \text{.....................(44)},$$

so that

$$\Omega = \log\left\{B\left[\frac{1+\sqrt{(1-w)}}{1-\sqrt{(1-w)}}\right]^{Ai}\right\} = \log\frac{dz}{dw},$$

and since $\Omega = 0$ when $w = 1$, and the argument of dz increases by $(\pi - \alpha)$ as w goes through zero, we find

$$\frac{dz}{dw} = \left[\frac{1+\sqrt{(1-w)}}{1-\sqrt{(1-w)}}\right]^{1-\alpha/\pi} \quad \text{..............(45).}$$

16. With our choice of constants the length of the part between the points corresponding to $w = 0$ and $w = 1$ is

$$\int_0^1 \left[\frac{1+\sqrt{(1-w)}}{1-\sqrt{(1-w)}}\right]^{1-\alpha/\pi} dw,$$

putting $w = \sin^2\theta$ and $\tan\frac{1}{2}\theta = x$, $\alpha/\pi = n$, we transform this into

$$8\int_0^1 \frac{x^{2n-1}(1-x^2)}{(1+x^2)^3}\,dx \quad \text{......................(46),}$$

and when α is an exact submultiple of π, n is an integer and we shall be able to evaluate the integral.

The pressure on the part between the same two points is

$$\tfrac{1}{2}\rho\int_0^1 \left[1-\left(\frac{dw}{dz}\right)^2\right]\frac{dz}{dw}\,dw,$$

or, making the same transformations as before,

$$4\rho\int_0^1 \frac{(x^{2n-1} - x^{3-2n})(1-x^2)}{(1+x^2)^3}\,dx \quad \text{...........(47).}$$

The above might be applied to find the pressure on the rudder of a ship when turned obliquely to the length of the ship. The average pressure, the centre of pressure, and the moment of the fluid pressures can all be expressed by means of definite integrals of similar form to the above. If it were worth while tables might be constructed giving the values of these quantities for any given inclination of the plane of the rudder to the longitudinal plane of the ship. As however the motion here considered is in two dimensions it is unlikely that the formulæ would yield any result of use in Navigation.

(4) *On thin rotating isotropic disks.* By C. CHREE, M.A., Fellow of King's College.

The following solution might be shortened by assuming certain results from a previous paper in the *Transactions*[1]. As the subject appears, however, to be one of considerable practical

[1] Vol. XIV. p. 328 et seq.

importance, I have on the advice of Professor Pearson made the proof complete in itself.

The term *disk* is here restricted to mean a thin plate of which a section parallel to the faces is bounded by a circle or two concentric circles. The disk is supposed of uniform density ρ and of an isotropic material, for which m and n are the elastic constants in the notation of Thomson and Tait's *Natural Philosophy.*

Taking the axis of the disk for axis of z with the origin O in the central plane—or plane bisecting the thickness of the disk—we see from the symmetry that as the disk rotates about its axis with uniform angular velocity ω the displacement at every point is in the plane through that point and the axis, having for its components w parallel to the axis and u along the perpendicular r on the axis directed outwards. At any point r, z in the disk the strains are as follows:

Normal strains.

longitudinal $\frac{dw}{dz}$ parallel Oz,

radial $\frac{du}{dr}$ along r,

transverse $\frac{u}{r}$ perpendicular to r and z.

Tangential or shearing strain.

$\frac{du}{dz} + \frac{dw}{dr}$ in the plane of zr.

It is obvious from the symmetry that the two other tangential strains must vanish.

The expression for the dilatation δ is

$$\delta = \frac{du}{dr} + \frac{u}{r} + \frac{dw}{dz} \quad \text{......................(1).}$$

The stress system is as follows:

$$\text{Normal stresses} \begin{cases} \widehat{zz} = (m-n)\,\delta + 2n\frac{dw}{dz} \text{ parallel to } oz, \\ \widehat{rr} = (m-n)\,\delta + 2n\frac{du}{dr} \text{ along } r, \\ \widehat{\phi\phi} = (m-n)\,\delta + 2n\,u/r \text{ perpendicular to } Oz \text{ and to } r; \end{cases}$$

$$\text{Shearing stress} \left\{ \widehat{rz} = n\left(\frac{du}{dz} + \frac{dw}{dr}\right) \text{ in plane } zr. \right.$$

The other two shearing stresses must vanish from the symmetry.

We may suppose the disk at rest, acted on by a "centrifugal force" $\omega^2\rho r$ per unit of volume. Thus the internal equations are the following two[1]

$$\frac{d\widehat{rr}}{dr}+\frac{d\widehat{rz}}{dz}+\frac{\widehat{rr}-\widehat{\phi\phi}}{r}+\omega^2\rho r=0 \quad\text{..............(2)},$$

$$\frac{d\widehat{rz}}{dr}+\frac{\widehat{rz}}{r}+\frac{d\widehat{zz}}{dz} \qquad = 0 \quad\text{..............(3)}.$$

Let $2l$ denote the thickness of the disk, a the radius of its outer a' that of its inner cylindrical surface, or *edge*. Then supposing the disk exposed to no surface forces, the solution ought to satisfy the following surface conditions[2]—

$$\text{over the flat faces } z=\pm l, \begin{cases}\widehat{zz}=0\text{..............(4)},\\ \widehat{rz}=0\text{..............(5)},\end{cases}$$

$$\text{over the edges } r=a \text{ and } r=a', \begin{cases}\widehat{rz}=0\text{..............(6)},\\ \widehat{rr}=0\text{..............(7)}.\end{cases}$$

Substituting for the stresses their expressions in terms of the strains and using (1), we easily transform (2) and (3) respectively into

$$(m+n)\,r\frac{d\delta}{dr}+n\frac{d}{dz}\left\{r\left(\frac{du}{dz}-\frac{dw}{dr}\right)\right\}=-\,\omega^2\rho r^2\text{........(8)},$$

$$(m+n)\,r\frac{d\delta}{dz}-n\frac{d}{dr}\left\{r\left(\frac{du}{dz}-\frac{dw}{dr}\right)\right\}=0 \quad\text{.............(9)}.$$

Differentiating (8) with respect to r and (9) with respect to z, then adding and dividing out by $(m+n)\,r$, we get

$$\frac{d^2\delta}{dr^2}+\frac{1}{r}\frac{d\delta}{dr}+\frac{d^2\delta}{dz^2}=-\frac{2\omega^2\rho}{m+n}\text{..............(10)}.$$

Of this a particular solution is

$$\delta=-\,\omega^2\rho r^2\div 2\,(m+n).$$

A complementary solution in ascending powers of r and z with arbitrary constants can be obtained, as I have shown in a previous paper[3]. Of this we require for our present purpose only the constant term and that of the second degree, or

$$\delta=A+C\,(z^2-\tfrac{1}{2}r^2).$$

[1] Pearson's *Elastical Researches of Barré de Saint-Venant*, foot-note, p. 79, or Ibbetson's *Mathematical Theory of Perfectly Elastic Solids*, p. 239.

[2] Ibbetson's *Mathematical Theory*...... l. c., or Todhunter and Pearson's *History of Elasticity*, Vol. I. Art. 614.

[3] *Transactions*, Vol. XIV. p. 328 et seq.

Putting the right-hand side of (10) zero, it is easily verified that this is a solution. We thus have

$$\delta = A + C\left(z^2 - \tfrac{1}{2}r^2\right) - \tfrac{1}{2}\omega^2\rho r^2/(m+n) \text{ ....(11).}$$

Noticing that

$$\frac{d}{dr}\left\{r\left(\frac{du}{dz} - \frac{dw}{dr}\right)\right\} = r\frac{d\delta}{dz} - r\frac{d^2w}{dz^2} - \frac{d}{dr}\left(r\frac{dw}{dr}\right),$$

we may transform (9) into

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \frac{d^2w}{dz^2} = -\frac{m}{n}\frac{d\delta}{dz} = -\frac{2m}{n}Cz \text{.........(12).}$$

Of this a particular solution is

$$w = -\frac{m}{3n}Cz^3.$$

A complementary solution is easily obtained in ascending powers of r and z. Of this we require for our present purpose only the terms of the first and third degrees, which separately satisfy (12) when the right-hand side is zero. Thus we get

$$w = \alpha_1 z + \epsilon_1(2z^3 - 3zr^2) - \frac{m}{3n}Cz^3 \text{ ....(13),}$$

where α_1 and ϵ_1 are new constants.

Employing (1) and (11), we in like manner easily transform (8) into

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} + \frac{d^2u}{dz^2} = -\frac{m}{n}\frac{d\delta}{dr} - \frac{\omega^2\rho r}{n},$$

$$= r\left(C\frac{m}{n} - \frac{\omega^2\rho}{m+n}\right) \text{...........(14).}$$

A particular solution is

$$u = \tfrac{1}{8}r^3\left(C\frac{m}{n} - \frac{\omega^2\rho}{m+n}\right).$$

For the complementary solution we require only the terms in odd powers of r and z up to the third degree. Terms in negative powers of z are of course inadmissible, and r^{-1} is the only negative power of r which satisfies the differential equation. Thus the complementary solution is

$$u = \frac{D}{r} + \alpha r + \epsilon(z^3 - 2zr^2) + \zeta(4z^2r - r^3),$$

where D, α, ϵ, ζ are new constants whose coefficients separately satisfy (14) when its right-hand side is 0. Thus for our complete solution of (14) we get

$$u = \frac{D}{r} + \alpha r + \epsilon(z^3 - 2zr^2) + \zeta(4z^2r - r^3) + \frac{1}{8}\left(C\frac{m}{n} - \frac{\omega^2\rho}{m+n}\right)r^3 \text{...(15).}$$

The constants in (11), (13) and (15) are not all arbitrary, being connected through the identity (1). From it we find

$$\alpha_1 = A - 2\alpha,$$

$$\epsilon_1 = \frac{1}{6}\frac{m+n}{n}C - \frac{4}{3}\zeta,$$

$$\epsilon = 0.$$

Thus the solution we have arrived at is

$$\left.\begin{aligned} \delta &= A + C(z^2 - \tfrac{1}{2}r^2) - \tfrac{1}{2}\omega^2\rho r^2/(m+n), \\ w &= (A - 2\alpha)z - \frac{m}{3n}Cz^3 + \frac{1}{6}\left(\frac{m+n}{n}C - 8\zeta\right)(2z^3 - 3zr^2), \\ u &= \frac{D}{r} + \alpha r + \zeta(4z^2r - r^3) + \frac{1}{8}\left(\frac{m}{n}C - \frac{\omega^2\rho}{m+n}\right)r^3 \end{aligned}\right\} \ldots(16),$$

where all the constants are independent.

To determine the constants we have the surface conditions (4)—(7).

From (4) and the expression for $\widehat{zz}$ in terms of the strains we easily find

$$(m+n)A - 4n\alpha + l^2\{(m+n)C - 16n\zeta\}$$
$$+ r^2\left\{8n\zeta - \frac{1}{2}(3m+n)C - \frac{1}{2}\frac{m-n}{m+n}\omega^2\rho\right\} = 0 \ldots(17).$$

Since this holds for all values of r between a' and a, the constant part and the coefficient of r^2 must separately vanish. Thus we get

$$(m+n)A - 4n\alpha + l^2\{(m+n)C - 16n\zeta\} = 0 \ldots\ldots(18),$$

$$8n\zeta - \frac{1}{2}(3m+n)C - \frac{1}{2}\frac{m-n}{m+n}\omega^2\rho = 0 \ldots\ldots\ldots(19).$$

From (5) and the expression for $\widehat{rz}$ in terms of the strains we at once obtain

$$16n\zeta - (m+n)C = 0 \ldots\ldots\ldots\ldots\ldots\ldots(20).$$

The equations (18), (19) and (20) are satisfied by

$$\left.\begin{aligned} \alpha &= \frac{m+n}{4n}A, \\ C &= -\frac{1}{2}\frac{m-n}{m(m+n)}\omega^2\rho, \\ \zeta &= -\frac{1}{32}\frac{m-n}{mn}\omega^2\rho \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(21).$$

Substituting these values in the expressions for the strains and stresses we find for all values of r and z

$$\left.\begin{aligned} \widehat{zz} &= 0, \\ \frac{du}{dz} + \frac{dw}{dr} &= 0, \\ \widehat{rz} &= 0 \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots(22).$$

and so

Since $\widehat{rz}$ is everywhere zero, the condition (6) over the edges is exactly satisfied. The only surface condition left is (7), but this we cannot *exactly* satisfy, unless $m = n$, by means of the present solution. For, substituting the above values of the arbitrary constants, we obtain from the expression for $\widehat{rr}$ in terms of the strains

$$\widehat{rr}_{r=a} = \tfrac{1}{2}(3m-n)A - 2na^{-2}D$$
$$-\frac{7m-n}{16m}\omega^2\rho a^2 - \frac{(m-n)(3m-n)}{4m(m+n)}\omega^2\rho z^2 \dots\dots(23),$$

$$\widehat{rr}_{r=a'} = \text{similar expression, replacing } a \text{ by } a' \quad \dots\dots\dots\dots(24).$$

It is obvious we cannot make these stresses vanish for all values of z.

If the thickness $2l$ of the disk be of the same order of magnitude as the radius a this failure renders the present method inapplicable; but when l/a is small it is easy to obtain a solution which according to Saint-Venant and other eminent authorities must be very approximately exact except in the immediate neighbourhood of the edges.

The principle this solution is based on is that of *statically equivalent* systems of loading. According to this principle when a surface of an elastic solid has a small dimension—such as the thickness of a thin disk—all systems of surface forces which in their distribution along the small dimension are statically equivalent produce, except in the immediate neighbourhood of their points of application, practically identical strains and stresses. We may thus for practical purposes replace any system of surface forces over the small dimension by any statically equivalent system.

For a discussion of this principle and illustrations of its application, the reader is referred to Saint-Venant's *Théorie de L'Élasticité...de Clebsch*, p. 174 et seq. and p. 727 et seq., also to Pearson's *Elastical Researches of Barré de Saint-Venant*, Arts. 8 and 9.

In my previous treatment of this problem[1], which was limited to a complete disk, I determined the constant A so as to make

[1] *Transactions*, Vol. xiv. pp. 334—5, § 76, first two cases; and *Quarterly Journal*, Vol. xxiii. 1889, pp. 24—28.

$\widehat{rr}_{r=a}$ vanish for a given value of z, viz. either $z=0$ or $z=\pm l$. In either case we are left with a system of unequilibrated normal forces along each generator of the edge. On this ground Professor Pearson has recently[1] expressed his opinion that my solution cannot be regarded as "final" even for a thin disk. As the normal forces in question are of the order of the *square* of the thickness of the disk, I am not altogether sure what weight may be attached to this criticism. In deference however to Professor Pearson's opinion, and to what I believe the view Saint-Venant would have taken, I propose the following method of solution which removes at least this objection.

It consists in determining A and D from the equations

$$\int_{-l}^{+l} \widehat{rr}_{r=a}\, dz = 0 \qquad (25),$$

$$\int_{-l}^{+l} \widehat{rr}_{r=a'}\, dz = 0 \qquad (26).$$

This still leaves normal stresses of the order of the square of the thickness over the edges, *but the forces along each generator of an edge form a system in statical equilibrium.* Thus according to the principle of statically equivalent systems, the solution we shall obtain—which must be strictly limited to thin disks—gives expressions for the strains and stresses which can differ sensibly from those supplied by the complete solution only in the immediate neighbourhood of the edges.

For the case of a complete disk D must vanish and A is to be determined by (25).

For the annular disk we find from (25) and (26)

$$\left.\begin{aligned} A &= \frac{7m-n}{8m(3m-n)}\,\omega^2\rho\,(a^2+a'^2) + \frac{m-n}{6m(m+n)}\,\omega^2\rho l^2, \\ D &= \frac{7m-n}{32mn}\,\omega^2\rho a^2 a'^2 \end{aligned}\right\} \qquad (27).$$

Also from equations (21) we have C and ζ determined explicitly, and α found in terms of A. Thus all the constants of our solution are determined. For a complete disk we have only to put $D=0$, and $a'=0$ in the expression for A in (27).

The physical results attainable from the solution will perhaps be rendered more practically serviceable by replacing the m, n of our previous work by Young's modulus E and Poisson's ratio η. To express the values obtained above for the arbitrary constants in terms of E and η we require the relations

$$\left.\begin{aligned} n &= \tfrac{1}{2}E/(1+\eta), \\ m &= \tfrac{1}{2}E/\{(1-2\eta)(1+\eta)\} \end{aligned}\right\} \qquad (28).$$

[1] *Nature*, 1891, p. 488.

Substituting the expressions found for the arbitrary constants in terms of E and η in (16), we find for the strains in an annular disk

$$\delta = \frac{\omega^2\rho(1-2\eta)}{4E}\{(3+\eta)(a^2+a'^2)-2(1+\eta)r^2\}$$
$$+\frac{\omega^2\rho}{3E}\frac{\eta(1-2\eta)(1+\eta)}{1-\eta}(l^2-3z^2) \quad \text{......(29)},$$

$$w = -\frac{\omega^2\rho}{4E}\eta\{(3+\eta)(a^2+a'^2)z-2(1+\eta)r^2z\}$$
$$-\frac{\omega^2\rho}{3E}\eta^2\frac{1+\eta}{1-\eta}z(l^2-z^2) \quad \text{......(30)},$$

$$u = \frac{\omega^2\rho}{8E}\left\{(1-\eta)(3+\eta)(a^2+a'^2)r-(1-\eta^2)r^3+\frac{a^2a'^2}{r}(1+\eta)(3+\eta)\right\}$$
$$+\frac{\omega^2\rho}{6E}\eta(1+\eta)r(l^2-3z^2) \quad \text{......(31)}.$$

From these strains we find for the stresses

$$\widehat{rr} = \frac{\omega^2\rho}{8}(3+\eta)\left\{a^2+a'^2-r^2-\frac{a^2a'^2}{r^2}\right\}$$
$$+\frac{\omega^2\rho}{6}\eta\frac{1+\eta}{1-\eta}(l^2-3z^2) \quad \text{......(32)},$$

$$\widehat{\phi\phi} = \widehat{rr}+\frac{\omega^2\rho}{4}\left\{(1-\eta)r^2+(3+\eta)\frac{a^2a'^2}{r^2}\right\} \quad \text{......(33)}.$$

For a complete disk we have

$$\delta = \frac{\omega^2\rho(1-2\eta)}{4E}\{(3+\eta)a^2-2(1+\eta)r^2\}$$
$$+\frac{\omega^2\rho}{3E}\eta\frac{(1-2\eta)(1+\eta)}{1-\eta}(l^2-3z^2) \quad \text{......(34)},$$

$$w = -\frac{\omega^2\rho}{4E}\eta\{(3+\eta)a^2z-2(1+\eta)r^2z\}$$
$$-\frac{\omega^2\rho}{3E}\eta^2\frac{1+\eta}{1-\eta}z(l^2-z^2) \quad \text{......(35)},$$

$$u = \frac{\omega^2\rho}{8E}(1-\eta)\{(3+\eta)a^2r-(1+\eta)r^3\}$$
$$+\frac{\omega^2\rho}{6E}\eta(1+\eta)r(l^2-3z^2) \quad \text{......(36)},$$

$$\widehat{rr} = \frac{\omega^2\rho}{8}(3+\eta)(a^2-r^2)+\frac{\omega^2\rho}{6}\eta\frac{1+\eta}{1-\eta}(l^2-3z^2) \quad \text{......(37)},$$

$$\widehat{\phi\phi} = \widehat{rr}+\frac{\omega^2\rho}{4}(1-\eta)r^2 \quad \text{......(38)}.$$

For both the annular and the complete disks $\widehat{zz}$ and $\widehat{rz}$ are by (22) everywhere zero. The expressions for the strains and stresses in a complete disk are correctly deduced from those in an annular disk by leaving out all terms containing a'^2. Allowing for the change of notation, the solution for the displacements in a complete disk differs from my previous one[1] only by terms in rl^2 in u, zl^2 in w and l^2 in δ. Thus it only adds to the strains given by the previous solution certain constant terms of order l^2, and in no respect modifies the conclusions derivable from that solution as to the mode in which the strains and stresses alter with the variables r, z.

The expressions (32) and (33) for the stresses in an annular disk when terms in l^2 and z^2 are neglected agree with those which Professor Ewing[2] quotes as obtained by Grossmann[3]. They likewise agree with those found by Clerk Maxwell[4] when the error in the sign of his equation (59) pointed out by Mr J. T. Nicolson[5] is corrected. The expression (31) for the radial displacement when terms of order l^2 are neglected is identical with that given implicitly or explicitly by Maxwell, and by Grossmann putting his $N_3 = 0$, and to the same degree of approximation (30) coincides with the value for the longitudinal displacement to which Maxwell's theory would lead if fully worked out. I shall thus for brevity speak of the expressions our solution supplies both for the complete and annular disks when terms of order l^2 are neglected as constituting the *Maxwell solution*.

The conclusion we are led to is that the methods of Maxwell and Grossmann—which seem practically identical—while involving inconsistencies[6] and certainly inconclusive from a strict theoretical standpoint, perhaps even "paradoxical" as Professor Pearson[7] states, yet lead to results which if the present investigation can be trusted are sufficiently exact for practical purposes so long as the disk is very thin.

From (33) it is obvious that $\widehat{\phi\phi}$ is everywhere greater than $\widehat{rr}$ in an annular disk. The same result follows from (38) for a complete disk, except at the axis where the two stresses are equal.

1 *Quarterly Journal of Pure and Applied Mathematics*, Vol. XXIII. 1889, *Equations* (129), p. 28.

2 *Nature*, 1891, p. 462.

3 *Verhandlungen des Vereins zur Beförderung des Gewerbfleisses*, Berlin, 1883, pp. 216—226.

4 *Transactions of the Royal Society of Edinburgh*, Vol. XX. Part I., 1853, pp. 111—112; or *Scientific Papers*, Vol. I. p. 61. For corrections to Maxwell's second equation (57) see Todhunter and Pearson's *History of Elasticity*, Vol. I. ft.-note, p. 827.

5 *Nature*, 1891, p. 514.

6 They lead to results inconsistent with one or both of the original assumptions, viz. that $\widehat{rz}$ is everywhere zero, and that $\widehat{zz}$ if not also zero is independent of r and z.

7 *Nature*, 1891, p. 488.

Also $\widehat{zz}$ and $\widehat{rz}$ vanish at every point, thus both in the annular and in the complete disk $\widehat{\phi\phi}$ is everywhere the *stress-difference* and u/r the *greatest strain*. Both quantities for any given value of r are greatest when $z = 0$, and for any given value of z are greatest when $r = 0$ for the complete disk, or $r = a'$ for the annular. They are thus according to the solution greatest in the central plane, at the centre of a complete disk and at the inner edge of an annular.

According, however, to the principle of statically equivalent surface forces our solution does not strictly apply for values of r which differ from a or from a' by quantities which do not exceed several multiples of l. In other words, it possibly may give values for the strains and stresses over the edges differing from the true values by terms of the order l^2. Thus in determining the greatest values of the stress-difference or greatest strain, which occur at or immediately adjacent to the inner edge in an annular disk, we are not warranted in retaining terms of this order of small quantities. I thus propose in determining these quantities to neglect terms of this order as being of doubtful accuracy, at least in an annular disk, and of insignificant magnitude in any thin disk. It should be noticed, however, that our complete solution gives at *all* radial distances larger values for the strains and stresses in the central plane than when terms in l^2 are neglected. Thus it would certainly only be prudent to regard the values we are about to find from the Maxwell solution for the maximum stress-difference and greatest strain as *minima*, which in all probability are exceeded in any actual case. In a thin disk, however, the true values can exceed these only by small terms, of order $(l/a)^2$ at least.

Neglecting then terms in l^2 and z^2, we find for the maximum stress-difference $\bar{S}$ and the largest value of the greatest strain $\bar{s}$—

$$\text{In a complete disk, } \begin{cases} \bar{S}_1 = \tfrac{1}{8}\omega^2\rho a^2(3+\eta) \ldots\ldots\ldots (39), \\ E\bar{s}_1 = (1-\eta)\bar{S}_1 \ldots\ldots\ldots\ldots (40), \end{cases}$$

$$\text{In an annular disk, } \bar{S}_2 = E\bar{s}_2 = \tfrac{1}{4}\omega^2\rho\,\{a^2(3+\eta) + a'^2(1-\eta)\} \ldots (41).$$

Since $\bar{S}_2 = Es_2$ the maximum stress-difference and greatest strain theories lead to identically the same result for the so-called "tendency to rupture"—i.e. approach to limit of linear elasticity—in the annular disk. In the complete disk the maximum stress-difference theory assigns for all possible values of η, except 0, a lower limit than the other for the safe velocity of rotation.

Supposing ω_1 and ω_2 the limiting safe angular velocities in a complete and in an annular disk of the same material and external radius, then putting in succession $\bar{S}_1 = \bar{S}_2$ and $\bar{s}_1 = \bar{s}_2$, we find

On the maximum stress-difference theory

$$\omega_1^2/\omega_2^2 = 2\left(1 + \frac{1-\eta}{3+\eta}\frac{a'^2}{a^2}\right) \quad\text{..............(42)},$$

On the greatest strain theory

$$\omega_1^2/\omega_2^2 = 2\,\frac{3+\eta+(1-\eta)\,a'^2/a^2}{(1-\eta)(3+\eta)} \quad\text{.........(43)}.$$

When $(a'/a)^2$ is negligible, or there is only a very small axial hole, these give respectively

$$\omega_1/\omega_2 = \sqrt{2} \text{ for all values of } \eta,$$

$$\omega_1/\omega_2 = \sqrt{2/(1-\eta)}, \text{ or nearly } 1{\cdot}633 \text{ for } \eta = {\cdot}25.$$

The former result was given by Professor Ewing in *Nature*, and he also directed attention to the great diminution it represents in the strength of a disk due to the removal of a small axial core. The effect is even more striking on the greatest strain theory for ordinary values of η.

Since the striking character of this result may arouse doubts in some minds as to the validity of any investigation which leads to it, I would point out that it is not an isolated fact. The removal of a central spherical core of any radius however small from a sphere rotating about a diameter has, as I have shown in a previous paper[1], a precisely similar effect, increasing very largely the greatest values both of the stress-difference and greatest strain. The same result also follows the removal of a thin axial core from a rotating right circular cylinder whose length is constrained to remain constant[2].

In discussing the nature of the strains and stresses we may for most purposes leave out of account in the first place terms of order l^2 or l^3, regarding them in the light of small corrections to the principal terms.

According to the Maxwell solution, every originally plane section parallel to the faces of a complete or annular disk approaches at every point the central section, $z = 0$, and assumes the form of a paraboloid of revolution about the axis of rotation. In this respect the phenomena are precisely similar to those presented by a flat oblate spheroid rotating about its axis of symmetry[3].

The latus rectum of the paraboloid into which is transformed

[1] *Transactions*, Vol. XIV. pp. 467—83. See Tables IV. and VIII. and their discussion.
[2] *Transactions*, Vol. XIV. p. 339.
[3] *Transactions*, Vol. XV. pp. 10—13.

an originally plane section at distance z from the central plane is in both the complete and annular disks

$$2E \div \{\eta (1+\eta) \omega^2 \rho z\}.$$

It thus depends merely on the angular velocity, the nature of the material and the original distance from the central plane. Its reciprocal, and so the curvature at the vertex of the paraboloid, varies directly as the square of the angular velocity and as the distance from the central plane.

The amount by which the axial point on an originally plane section parallel to the faces—of course an imaginary point in an annular disk—approaches the central plane varies as the square of the radius of the complete disk. The corresponding approach in an annular disk varies as the sum of the squares of the radii of its edges, and is greater than in a complete disk of the same external radius. The magnitudes of the reductions in the thickness, $2l$, of the disks will be seen from the following data—

$$\text{In annular disk} \begin{cases} \text{at inner edge } \dfrac{1}{2E}\omega^2 \rho l \eta \{(3+\eta) a^2 + (1-\eta) a'^2\}, \\ \text{at outer edge } \dfrac{1}{2E}\omega^2 \rho l \eta \{(3+\eta) a'^2 + (1-\eta) a^2\}, \end{cases}$$

$$\text{In complete disk} \begin{cases} \text{at axis } \dfrac{1}{2E}\omega^2 \rho l \eta (3+\eta) a^2, \\ \text{at outer edge } \dfrac{1}{2E}\omega^2 \rho l \eta (1-\eta) a^2. \end{cases}$$

The terms in zl^2 and z^3 in (30) and (35) cut out when $z = \pm l$, so that as regards the preceding results as to the change of thickness there is an exact agreement between the Maxwell solution and the more complete solution.

It will be noticed that the reduction in thickness at the inner edge of an annular disk equals the reduction at the axis of a complete disk equal in radius to its outer edge, together with the reduction at the rim of a complete disk equal in radius to its inner edge, the thicknesses, materials and angular velocities being the same in each case. Also the reduction in thickness at the outer edge of an annular disk exceeds what it would be if the disk were complete by the reduction at the axis of a complete disk equal in radius to its inner edge.

The longitudinal strain dw/dz parallel to the axis is a compression at every point, unless $\eta = 0$, both in the complete and annular disks, and diminishes numerically as r increases. For ordinary materials it is a quantity of the same order of magnitude as the radial and transverse strains, but it vanishes throughout if $\eta = 0$.

The terms in zl^2 and z^3 in w would indicate that the longitudinal compression is somewhat greater near the central plane and somewhat less near the faces of the disk than according to the Maxwell solution.

For a first approximation confining our attention to the Maxwell solution in (31) and (36), we see that every point in the disk, whether complete or annular, increases its distance from the axis, and the transverse strain u/r is thus everywhere an extension.

In the complete disk the radial strain is an extension inside and a compression outside of a cylindrical surface co-axial with the disk, and of radius r_1, given by

$$r_1 = a\sqrt{\tfrac{1}{3}(3+\eta)/(1+\eta)} \quad\text{..............(44).}$$

This gives a value for r_1 less than a for all possible values of η except 0. Any annulus of the disk increases or diminishes in radial thickness according as it lies inside or outside this surface.

In the annular disk the increases ∂a and $\partial a'$ in the radii of the edges, and $\partial(a-a')$ in the *radial thickness*, $a-a'$, are given by

$$\partial a = \frac{\omega^2\rho a}{4E}\{(1-\eta)a^2 + (3+\eta)a'^2\} \quad\text{.........(45),}$$

$$\partial a' = \frac{\omega^2\rho a'}{4E}\{(3+\eta)a^2 + (1-\eta)a'^2\} \quad\text{.........(46),}$$

$$\partial(a-a') = \frac{\omega^2\rho(a-a')}{4E}\{(a-a')^2 - \eta(a+a')^2\} \quad\text{..(47).}$$

Thus the radial thickness is increased or diminished according as

$$a'/a < \text{ or } > (1-\sqrt{\eta}) \div (1+\sqrt{\eta}) \quad\text{............(48).}$$

For the ratio of the radii in the annular disk whose radial thickness is unaltered, we find

$$\eta = 0, \qquad a'/a = 1,$$
$$\eta = \cdot 25, \qquad a'/a = \cdot\dot{3},$$
$$\eta = \cdot 36, \qquad a'/a = \cdot 25,$$
$$\eta = \cdot 5, \qquad a'/a = \cdot 1716 \text{ approx.}$$

The radial strain given by the Maxwell solution in (31) is

$$\frac{du}{dr} = \frac{\omega^2\rho}{8E}f(r) \quad\text{........................(49),}$$

where $f(r) = (1-\eta)(3+\eta)(a^2+a'^2) - 3(1-\eta^2)r^2$
$$- (1+\eta)(3+\eta)a^2a'^2r^{-2} \quad\text{...(50).}$$

Since $\qquad f(a) = -2\eta\{(1-\eta)a^2 + (3+\eta)a'^2\},$

and $\qquad f(a') = -2\eta\{(1-\eta)a'^2 + (3+\eta)a^2\},$

we see that the radial strain is a compression at both edges for all ratios of $a' : a$, and for all possible values of η except 0. For $\eta = 0$,

$$f(r) = \frac{3}{r^2}(r^2 - a'^2)(a^2 - r^2) \quad \text{..............(51)},$$

and so the radial strain is for all ratios of $a' : a$ an extension everywhere except at the edges, where it vanishes.

For all other values of η there are always portions of the disk immediately adjacent to the edges wherein the radial strain is a compression, and it is easily proved that the radial strain is everywhere a compression when

$$a'/a > \sqrt{(1-\eta)(3+\eta)} \div \{3^{\frac{1}{2}}(1+\eta) + \sqrt{4\eta(2+\eta)}\} \ldots (52).$$

An idea of the nature of the radial strain under various conditions for the value ·25 of η may be derived from the following table:

TABLE I.

Sign of $\frac{du}{dr}$, and loci where it vanishes for $\eta = \cdot 25$.

Value of a'/a			$\frac{du}{dr}$				
		−	0	+	0	−	
·1	$r/a =$ ·1		·130		·927		1
·2	·2		·264		·912		1
·3	·3		·409		·882		1
·4	·4		·597		·806		1
> ·426	Everywhere a compression.						

From the Maxwell solution in (32) and (37) it is obvious that the radial stress is a traction for all possible values of η at all points not on the edge or edges of the disk.

The terms in rl^2 and rz^2 in (31)—(33) and (36)—(38) show that the complete solution gives algebraically greater values for the radial and transverse displacements, strains and stresses at all axial distances in the central plane than the Maxwell solution. It will be noticed, however, that the mean of each of these

quantities taken between the limits $-l$ and $+l$ of z is the same as when terms of order l^2 are neglected. Thus the Maxwell-Grossmann method of solution leads to values for all the radial and transverse strains and stresses which are identical with those which the present solution supplies for mean values taken throughout the thickness of the disk. It also as we have seen when fully worked out supplies the same value for w at the plane surfaces of the disk, and so the same mean value for the longitudinal compression throughout the thickness.

In consequence of the second relation (22) the mutual inclinations of all material lines in the disk remain unchanged. Thus what were originally cylindrical surfaces co-axial with the original edges cut orthogonally the surfaces into which have been transformed what were originally planes parallel to the faces; i.e. they become orthogonal to what are practically a series of paraboloids, whose common axis is that about which the rotation takes place. This perhaps will convey the clearest idea of how material lines originally perpendicular to the faces become under rotation concave to the axis of the disk.

May 18, 1891.

PROFESSOR LIVEING, VICE-PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:

(1) *On Parasitic Mollusca.* By A. H. COOKE, M.A., King's College.

[*Received July* 18, 1891.]

Various grades of parasitism occur among the Mollusca, from the true parasite, living and nourishing itself on the tissues and secretions of its host, to simple cases of commensalism. Some authors have divided these forms into endo- and ecto-parasites, according as they live inside or outside of their host. Such a division, however, is hardly tenable. Certain forms are indifferently endo- and ecto-parasitical, while others are ecto-parasitic in the young form, and become endo-parasitic in the adult. It will be convenient therefore, simply to group the different forms according to the home on which they find a lodgement.

On *Cœlenterata*. (*a*) Sponges. *Vulsella* and *Crenatula* almost invariably occur in large masses of irregular shape, boring into sponges. (*b*) Corals. These form a favourite home of many species, amongst which are several forms of *Coralliophila*, *Rhizochilus*, *Leptoconcha*, and *Sistrum*. The common *Magilus*, from the Red Sea and Indian Ocean, in the young form is shaped like a

small *Buccinum*. As the coral (*Meandrina*) to which it attaches itself grows, it develops at the mouth a long calcareous tube, the aperture of which keeps pace with the growth of the coral, and prevents the mollusc from being entombed. The animal lives at the free end of the tube, and is thus continually shifting its position, while the space it abandons becomes completely closed by calcareous matter. Certain species of *Ovula* inhabit *Gorgoniae*, assuming the colour, yellow or red, of their host, and, in certain cases, developing for prehensile purposes a pointed extension of the two extremities of the shell. *Pedicularia* inhabits the common *Melithaea rubra* of the Mediterranean, and another species has been noticed by Graeffe[1] on *M. ochracea* in Fiji.

On *Echinodermata*. (*a*) *Crinoidea*. *Stilina comatulicola* lives on *Comatula mediterranea*, fixed to the outer skin, which it penetrates by a very long proboscis; the shell is quite transparent[2]. A curious case of a fossil parasite has been noticed by Roberts[3]. A *Calytraea*-shaped shell named *Platyceras* always occurred on the ventral side of a crinoid, encompassed by the arms. For some time this was thought to afford conclusive proof of the rapacity and carnivorous habits of the echinoderm, which had died in the act of seizing its prey. Subsequent investigations, however, showed that in all the cases noticed (about 150) the *Platyceras* covered the anal opening of the crinoid in such a way that the mouth of the mollusc must have been directly over the orifice of the anus. (*b*) *Asteroidea*. The comparatively soft texture of the skin of the starfishes renders them a favourite home of various parasites. The brothers Sarasin noticed[4] a species of *Stilifer* encysted on the rays of *Linckia multiformis*. Each shell was enveloped up to the apex, which just projected from a hole at the top of the cyst. The proboscis was long, and at its base was a kind of false mantle, which appeared to possess a pumping action. On the under side of the rays of the same starfish occurred a capuliform mollusc (*Thyca ectoconcha*), furnished with a muscular plate, whose cuticular surface was indented in such a way as to grip the skin of the *Linckia*. This plate was furnished with a hole, through which the pharynx projected into the texture of the starfish, acting as a proboscis and apparently furnished with a kind of pumping or sucking action. Adams and Reeve[5] describe *Pileopsis astericola* as living 'on the tubercle of a starfish,' and *Stilifer astericola*, from the coast of

[1] Described as a *Cypraea*, but no doubt an *Ovula* or *Pedicularia*: C. B. Bakt. Par. v. 543.

[2] Von Graff, *Z. Wiss. Zool.* xxv. 124.

[3] *Proc. Amer. Phil. Soc.* xxv. 231.

[4] Ergeb. naturw. Forsch. Ceylon, abstr. in *J. R. M. S.* (2) vi. 412.

[5] *Voyage of the Samarang*, Moll. p. 69, Pl. xi. f. 1; p. 47, Pl. xvii. f. 5.

Borneo, as 'living in the body of a starfish.' In the British Museum there is a specimen of *Pileopsis crystallina* in situ on the ray of a starfish. On the brittle starfishes (*Ophiuridae*) occur several species of *Stiliferina*. (*c*) *Echinoidea*. Various species of *Stilifer* occur on the ventral spines of echinoderms, and are sometimes found imbedded in the spines themselves. *St. Turtoni* occurs on the British coasts on several species of *Echini*, and *Montacuta substriata* frequents *Spatangus purpureus* and certain species of *Amphidetus*, *Cidaris* and *Brissus*. *Lepton parasiticum* has been described from Kerguelen I. on a *Hemiaster*, and a new genus, *Robillardia*, has recently been established[1] for a *Hyalinia*-shaped shell, parasitic on an *Echinus* from Mauritius. (*d*) *Holothuroidea*. The 'sea-cucumbers' afford lodgement to a variety of curious forms, some of which have experienced such modifications that their generic position is by no means established. *Entoconcha* occurs fixed by its buccal end to the blood-vessels of certain *Synaptae* in the Mediterranean and the Philippines. *Entocolax* has been dredged from 180 fath. in Behring's Straits, attached by its head to certain anterior muscles of a *Myriotrochus*[2]. A curious case of parasitism is described by Voeltzkow[3] as occurring on a *Synapta* found between tide-marks on the I. of Zanzibar. In the oesophagus of the *Synapta* was found a small bivalve (*Entovalva*), the animal of which was very large for its shell, and almost entirely enveloped the valves by the mantle. As many as five specimens occurred on a single *Synapta*. In the gut of the same holothurian lived a small univalve, not creeping freely, but fixed to a portion of the stomach wall by a very long proboscis which pierced through it into the body cavity. This proboscis was nearly three times as long as the animal, and the forward portion of it was set with sharp thorns, no doubt to make it to retain its hold and resist evacuation. Various species of *Eulima* have been noticed in every part of the world, from Norway to the Philippines, both inside and outside Holothurians[4]. *Stilifer* also occurs on this section of echinoderms[5].

On *Annelida*. *Cochliolepas parasiticus* has been noticed under the scales of *Acoetes lupina* (a kind of 'sea-mouse') in Charleston Harbour[6].

On *Crustacea*. A mussel, $\frac{3}{8}$ in. long, has been found[7] living under the carapace of the common shore-crab (*Carcinus maenas*), but this is not so much a case of parasitism as of involuntary

[1] E. A. Smith, *Ann. Mag. Nat. Hist.* 1889 (i), 270.
[2] *Journ. de Conch.* (3) xxix. 101.
[3] *Zool. Jahrb. Abth. f. Syst.* v. 619.
[4] See especially Semper, *Animal Life*, Ed. 1, p. 351.
[5] Gould, *Moll. of U.S. expl. exped.* 1852, p. 207 (*St. acicula*, from Fiji).
[6] Stimpson, *Proc. Bost. Soc. N. H.* vi., 1858, p. 308.
[7] Pidgeon, *Nature*, xxxix. p. 127.

habitat, the mussel no doubt having become involved in the branchiae of the crab in the larval form.

On *Mollusca.* A species of *Odostomia* (*pallida* Mont.) is found on our own coasts on the 'ears' of *Pecten maximus*, and also[1] on the operculum of *Turritella communis.* At Panama the present writer found *Crepidula* (2 sp.) plentiful on the opercula of the great *Strombus galea* and of *Cerithium irroratum.* *Amalthea* is very commonly found in *Conus, Turbo,* and other large-sized shells, but this is probably not a case of parasitism, but simply of convenience of habitat, just as young oysters are frequently seen on the carapace and even on the legs of large crabs.

On *Tunicata.* *Lamellaria* is said to deposit its eggs on an Ascidian (*Leptoclinum*), and the common *Modiolaria marmorata* lives in colonies imbedded in the tegument of *Ascidia mentula* and other simple Ascidians.

Special points of interest with regard to parasitic mollusca relate to (1) *Colour.* This is in most cases absent, the shell being of a uniform hyaline or milky white. This may be due, in the case of the endo-parasitic forms, to absence of light, and possibly, in those living outside their host, to some deficiency in the nutritive material. A colourless shell is not necessarily protective, for though a transparent shell might evade detection, a milk-white hue would probably be conspicuous. (2) *Modifications of structure.* These are in many cases considerable. *Entoconcha* and *Entocolax* have no respiratory or circulatory organs and no nervous system; *Thyca* and certain *Stiliferi* possess a curious suctorial apparatus; the foot in many cases has aborted, since the necessity for locomotion is reduced to a minimum[2], and its place is supplied by an enormous development of the proboscis, which enables the creature to provide itself with nutriment without shifting its position. Special provision for holding on is noticed in certain cases, reminding us of similar provision in human parasites. Eyes are frequently, but not always wanting, even in endo-parasitic forms. A specially interesting modification of structure occurs in (3) the *Radula.* In most cases (*Eulima, Stilifer, Odostomia, Entoconcha, Entocolax, Magilus, Coralliophila, Leptoconcha*) it is absent altogether. In *Ovula* and *Pedicularia,* genera which are in all other respects closely allied to *Cypraea,* the radula exhibits marked differences from the typical radula of the *Cypraeidae.* The formula (3·1·3) remains the same, but the laterals are greatly produced and become fimbriated, sometimes at the extremity only, sometimes along the whole length. A

[1] Smart, *Journal of Conch.* v. 152.

[2] Semper notices a case where a *Eulima* whose habitat is the stomach of a Holothurian retains the foot unmodified, while a species occurring on the outer skin, but provided with a long proboscis, has lost its foot altogether.

very similar modification occurs in the radula of *Sistrum spectrum* Reeve, a species which is known to live parasitically on one of the branching corals. Here the laterals differ from those of the typical *Purpuridae* in being very long and curved at the extremity. The general effect of these modifications appears to be the production of a radula rather of the type of the vegetable-feeding *Trochidae*, which may perhaps be regarded as a link in the chain of gradually degraded forms which eventually terminate in the absence of the organ altogether. The softer the food, the less necessity there is for strong teeth to tear it; the teeth either become smaller and more numerous, or else longer and more slender, and eventually pass away altogether. It is curious, however, that the same modified form of radula should appear in species of *Ovula* (e.g. *ovum*), and that the same absence of radula should occur in species of *Eulima* (e.g. *polita*) known not to be parasitical. This fact perhaps points back to a time when the ancestral forms of each group were parasitical and whose radulae were modified or wanting, the modification or absence of that organ being continued in some of their non-parasitical descendants.

(2) *Exhibition of models of double supernumerary appendages in Insects: also of a mechanical method of demonstrating the system upon which the Symmetry of such appendages is usually arranged.* By W. BATESON, M.A., St John's College.

(3) *On the nature of the excretory processes in Marine Polyzoa.* By S. F. HARMER, M.A., King's College.

[*Abstract: reprinted from the Cambridge University Reporter, May* 26, 1891.]

This communication was the result of an occupation of a University table at the Zoological Station at Naples during the Easter Vacation of 1891.

Observations were made on the manner in which various artificial pigments were excreted in *Bugula* and in *Flustra*, on the lines adopted by Kowalevsky (*Biolog. Centralblatt*, IX., 1889—1890, pp. 33 etc.) for other Invertebrates. The general result of the experiments was to show that excretion is not performed by organs comparable with nephridia, but that this process is carried on by free mesoderm cells, and to some extent by the connective tissue and by the walls of the alimentary canal. Evidence was obtained to show that the periodic loss of the alimentary canals leading to the formation of the "brown bodies" may be regarded as, to some extent, an excretory process.

June 1, 1891.

PROF. G. H. DARWIN, PRESIDENT, IN THE CHAIR.

The following communications were made to the Society:

(1) *On the part of the parallactic class of inequalities in the moon's motion, which is a function of the ratio of the mean motions of the sun and moon.* By ERNEST W. BROWN, M.A., Fellow of Christ's College.

In a paper to be published shortly, a solution will be given by approximation in series, of the equations for this class of inequalities. In Vol. I. of the *American Journal of Mathematics*, Mr G. W. Hill has shown that by using rectangular instead of polar co-ordinates, the inequalities depending *only* on the mean motions of the sun and moon can be found to a high degree of accuracy with comparatively little trouble; and that the series to be obtained may be rendered more convergent by developing in terms of $n'/(n-n') = m'$, instead of $n'/n = m$ as has been done in most of the previous theories. He further shows that by developing in terms of $m'/(1-\frac{1}{3}m') = \mu$ a still greater degree of convergency is obtained. The results are expressed in rectangular co-ordinates and on that account are not convenient for obtaining the algebraical expressions of the longitude and parallax of the moon. But these transformations will still have force when we change to polar co-ordinates, so that by putting in Delaunay's series for this class of inequalities $m = m'/(1+m') = \mu/(1+\frac{4}{3}\mu)$ and expanding in powers of m' or μ we should get a better approximation to the truth. This is not necessary in the Variation Inequality which Delaunay has found with sufficient accuracy for practical purposes. But in the Parallactic Inequality he stops at $(a/a').m^7$, the numerical multiplier of which is roughly 55113; the term expressed in seconds is $0''\cdot38$. By substituting

$$m = m'/(1+m')$$

and developing in terms of m', this multiplier is reduced to about one half its former value; but since m' is nearly one-twelfth greater than m the accuracy is not increased, though the new series has greater convergency.

On using Hill's method with rectangular co-ordinates for the parallactic class of inequalities (i.e. those dependent on the ratio of the mean distances of the moon and sun) a factor whose value is $1/(1-4m'-\ldots)$ appears, and to the expansion of this factor in powers of m' is due the slow convergence of the series giving the

coefficient of the parallactic inequality. Delaunay's series for this inequality is[1]

$$-\frac{a}{a'}\Big[\underset{74''\cdot02}{\tfrac{15}{8}m} + \underset{34''\cdot33}{\tfrac{93}{8}m^2} + \underset{11''\cdot89}{\tfrac{6887}{128}m^3} + \underset{4''\cdot43}{\tfrac{137197}{512}m^4} + \underset{1''\cdot86}{\tfrac{4628333}{3072}m^5} + \underset{0''\cdot71}{\tfrac{63106813}{8192}m^6} + \underset{0''\cdot38}{\tfrac{10835537159}{196608}m^7}\Big],$$

where the coefficients expressed in seconds of arc are written below. In this expression put $m = m'/(1+m')$ and expand in powers of m', subtract from the result the expansion of

$$(\tfrac{15}{8}m' + \tfrac{9}{4}m'^2)/(1-4m')$$

(which are the first two terms found by rectangular co-ordinates) in powers of m' and we obtain instead of Delaunay's series the expression

$$-\frac{a}{a'}\Big[\underset{-118''\cdot25\ \ -11''\cdot47}{(\tfrac{15}{8}m' + \tfrac{9}{4}m'^2)/(1-4m')} - \underset{+1''\cdot83}{\tfrac{841}{128}m'^3} - \underset{+0''\cdot37}{\tfrac{8423}{512}m'^4} + \underset{-0''\cdot16}{\tfrac{273317}{3072}m'^5} - \underset{+0''\cdot02}{\tfrac{3140657}{24576}m'^6} + \underset{-0''\cdot20}{\tfrac{3320080247}{196608}m'^7}\Big].$$

Newcomb[2] suggests that the last term of Delaunay's expression is wrong and this expression which is deduced from Delaunay's would seem to support that view. I hope however to verify the term. Leaving this term out of consideration the increased convergency of the series is manifest. The first three terms give nearly the whole value of the coefficient. In the paper the other coefficients of the periodic inequalities of this class will be dealt with in a similar manner, and expansions will be given for the whole class of inequalities, the factor $1/(1-4m'-\ldots)$ being introduced also into the higher powers of m'. It will then not be necessary to go further than m'^5a/a' to get the values of the coefficients correct to one hundredth of a second of arc; and, for this degree of accuracy, by the methods given the approximations, either for algebraical or numerical results, are not long.

(2) *On Pascal's Hexagram.* By H. W. RICHMOND.

The author applies Cremona's method of deriving the hexagram by projection of the lines on a nodal cubic surface from the node. By use of a new form of the equation to this surface the equations of the lines are obtained in a perfectly symmetrical form, and their properties thence developed.

[1] *Mémoires de l'Académie des Sciences*, Tome XXIX. p. 847.
[2] *Astronomical Papers for use of American Ephemiris*, Vol. I. pt. II. p. 71.

(3) *A Linkage for describing Lemniscates and other Inverses of Conic Sections.* By R. S. COLE.

(4) *Some experiments on liquid electrodes in vacuum tubes.* By C. CHREE, M.A., Fellow of King's College.

The experiments discussed in this paper were undertaken at the suggestion of Professor J. J. Thomson, in order to throw further light on the nature of the electric discharge in a vacuum tube with liquid electrodes. To render the work intelligible it is necessary to give a brief sketch of the general nature of the discharge at low gaseous pressures, and to mention certain results of previous observers and certain of their theoretical conclusions.

At low pressures the phenomena at the cathode—or electrode to which the positive current travels in the tube—are the most striking, so that it is the most convenient point of departure. The phenomena ordinarily observed between the electrodes when not too near together are as follows:—

(A) a thin luminous envelope covering the cathode,

(B) a much thicker, but still except at very low pressures, short dark space,

(C) a bright, usually blue, space of considerable length,

(D) a second dark space,

(E) a more or less luminous interval extending to the anode.

There is unfortunately no universally accepted English terminology for these spaces. The following table shows some of the terms most commonly used in English, and also the ordinary German terminology.

TABLE I.

	GERMAN	SCHUSTER	SPOTTISWOODE AND MOULTON	ORDINARY USAGE
(A)	Lichtsaum	Narrow layer		
(B)	dunkle Kathodenraum	Dark space	Crookes' space	Crookes' space
(C)	Glimmlicht {helle Kathodenschicht {Glimmlichtstrahlen	Glow proper	{Negative glow {Negative haze	
(D)	dunkle Trennungsraum	Dark interval	Negative dark space	Faraday space
(E)	positive Licht	Positive light	Positive light	

(A), (B) and (C) are regarded as forming the negative light or discharge, and (D) as separating the negative and positive discharges. (A) is very inconspicuous, if actually existent, at gaseous pressures exceeding 1 or 2 mm. of mercury, but at very low pressures it is fairly bright though very thin.

(B) is also insignificant so long as the pressure exceeds a few mm. of mercury, but at very low exhaustions it has been observed to exceed a length of 2 cm. Its length has been shown not to depend much on the material of the cathode when metallic. It is also usually but little dependent on the strength of the current. It varies to a considerable extent with the nature of the gas, being according to Professor Crookes[1] decidedly longer in hydrogen and shorter in carbonic acid gas than in air. In any one gas it is supposed to increase in length as the pressure is reduced, so that its magnitude gives a useful if not very exact indication of the degree of exhaustion[2]. The following table gives some of the measurements of Crookes[3]—altered to mm.—and Puluj[4] for air vacua.

TABLE II.

Crookes.		Puluj.	
Pressure in mm. of Hg.	Length of (B) in mm.	Pressure in mm. of Hg.	Length of (B) in mm.
·313	6·5	1·46	2·5
·123	8·5	·66	4·5
·078	12·0	·30	7·8
·042	15·0	·24	9·5
·020	25	·16	14·0
		·06	22·0

For a given length of (B) the pressures found by Crookes are very considerably less than those found by Puluj, a discrepancy ascribable perhaps to differences between their tubes and cathodes but due probably in greater measure to the uncertainties attending the determination of such low pressures.

The conditions of Puluj's experiments resembled more closely those of the present paper than did Crookes', so the former's results seem *a priori* the best for comparison with those described here. It must, however, be remembered that at the lowest pressures in my experiments, the gas in the tube was doubtless in great measure, if not almost exclusively, vapour from the liquid electrodes. Though not absolutely black, (B) in

[1] *Phil. Trans.* 1879, pp. 138—9.
[2] *Phil. Trans.* 1879, p. 137.
[3] *Phil. Trans.* 1879, pp. 158—9.
[4] *Sitzungsberichte Math. Nat. Classe der k. Akad.*, Bd. LXXXI., Abth. II. Wien, 1880, p. 874.

general seems so to the eye, and the surface separating it from (C) is usually sharply defined.

(C) has its brightest side next the cathode, and it is by some writers divided into a brighter portion, the negative glow, and a less bright portion, the negative haze. (C) appears almost as soon as the pressure is sufficiently reduced to allow the discharge to pass. It increases in length[1] as the pressure is reduced within at least certain limits. The transition from (C) to (D) is usually so gradual no exact line of separation can be drawn. (D) is not always visible. Its presence depends to a considerable extent on the strength of the discharge. At low pressures an increase in the strength of discharge tends to make (D) contract[2]. At high pressures when the discharge first passes (E) consists of a succession of twig-like independent discharges, which, as the pressure is lowered, transform into what seems to the eye a tolerably uniform column whose colour in most gases, especially in air, is bright red. As the exhaustion proceeds the colour becomes less bright, and the apparent uniformity of the column tends to disappear. At moderate pressures such as 1 or 2 mm. of mercury with a regular source of current, (E) consists in general of a succession of similar units, striae, each having a sharply defined luminous head on the side next the cathode, and gradually fading on the side next the anode into a seemingly non-luminous portion. At very low pressures the striae are sometimes very faintly defined, if existent, and (E) appears as a hazy luminosity, generally of a blue tint.

Taking the simplest case, viz. a long, uniform, straight cylindrical tube with flat metal electrodes, whose planes are perpendicular to the axis of the tube, and whose diameters are not very small, each stria in the positive column has in anatomical language an opisthocoelous form, the convex surface being that of the luminous head. This surface is, however, in general of smaller curvature than a sphere of diameter equal to that of the tube. Whether the positive column be striated or not, the surface separating it from (D) has its convexity on the side next the cathode. This surface is usually sharply defined when (D) appears at all.

There is another phenomenon whose relation to the discharge is somewhat doubtful, viz. the phosphorescence observed in good vacua. The more brilliant phosphorescent phenomena are beautifully shown by many well-known experiments of Professor Crookes. He believes the "molecular streams" or "radiant matter"—German "Kathodenstrahlen"—whose incidence on the

[1] After completely covering a wire cathode it expands as the current increases. See Hittorf, Wied. *Ann.* 20, 1883, p. 746.

[2] Hittorf, l. c. pp. 736—7.

glass of the tube sets up this luminosity, to be gas molecules charged at the cathode and projected with great velocity at right angles to its surface. According to Goldstein and others the direction of projection varies to some extent from the normal. Professor Crookes[1] appears to have originally thought these molecular streams peculiar to very low vacua and indicative of a fourth state of matter, bearing to the gaseous state somewhat the same relation as it bears to the liquid. Messrs Spottiswoode and Moulton[2] have, however, shown that phosphorescence can be produced at quite high pressures provided the intensity of the negative discharge be sufficiently increased.

These latter observers[3] have treated with great fulness other less conspicuous phosphorescent effects. They found that under certain conditions a portion of the wall of a vacuum tube touched by the finger or having an earthed conductor in its neighbourhood acts as a sort of secondary cathode, setting up phosphorescence on the opposite side of the tube. They also found that at very low pressures the positive discharge when in the form of a hazy luminous column, occupying in general only a portion of the tube's cross-section, creates a sort of demand for negative electricity which may be supplied by a discharge proceeding from the walls of the tube in directions at right angles to its length and creating phosphorescence. Further when the positive column is cut at an angle by the wall of the tube, as at a sharp bend, phosphorescence appears whose position is as if it were due to the impact of molecules travelling along the positive column in the direction from cathode to anode. This latter phenomenon has been more exhaustively treated by Goldstein[4]. He found that at very low pressures in a tube bent at right angles any number of times, there is at every bend between the cathode and anode where the positive column extends, a patch of phosphorescence, situated as if due to rays travelling from the cathode to the anode. Supposing an electrode at A in a tube AD at right angles to a tube BDE, and that the latter tube is closed at the end B, while a third tube at right angles to it leads from E to an electrode C, then according to Goldstein, if I interpret him correctly, there is phosphorescence at E whether the cathode be A or C, but none at B unless C is cathode.

Goldstein[5] also describes the production of phosphorescence at positions which could not be reached by molecular streams travelling in straight lines from the cathode, which he apparently

[1] *Phil. Trans.* 1879, pp. 142, 143, 163, 164, etc.
[2] *Phil. Trans.* 1880, pp. 582—6, etc.
[3] l. c., pp. 602—6, 616—7, 620, etc.
[4] *Wied. Ann.* 12, 1881, pp. 104—9, and figs. 16 and 17, Taf. I.
[5] *Wied. Ann.* 15, 1882, pp. 246—254.

ascribes to a species of diffuse reflexion at a bend in a narrow part of the tube. Perhaps this is immediately connected with another phenomenon he observed[1], viz. that if the vacuum tube be of variable section then a place where in passing from cathode to anode there is a sudden large increase in the diameter takes upon itself the functions of a secondary cathode, dominating the character of the striae on the side next the anode and emitting molecular streams along the axis of the tube.

The view that phosphorescence is due to the impact of finely divided matter torn off the cathode by the current has been maintained by Puluj[2] and others, but it seems obviously inapplicable to the phosphorescence proceeding from a secondary cathode, or to that called in by the positive column. Even as concerns the ordinary phenomena some experiments devised by Crookes[3] seem very adverse to Puluj's view. It is, however, unquestionable that cathodes of most substances have matter torn off by the discharge, and that it is frequently largely deposited on those portions of the tube where the ordinary phosphorescence is most conspicuous. Messrs Spottiswoode and Moulton[4] have pointed out other resemblances between the action of the molecular streams and that of matter such as lamp-black actually projected from the cathode.

As regards the function of the molecular streams in the discharge no final results have been obtained, though much speculation exists. Messrs Spottiswoode and Moulton give as the result of their investigations:—"At present we have come to no definite conclusion..., but we cannot say that we are aware of anything that conclusively shows that they (the molecular streams) have any definite electrical function to perform in the discharge," l. c. p. 650. "The most attractive hypothesis relating to their functions is that they officiate at the birth of the discharge and enable it to get into the gaseous medium...," l. c. p. 651.

From the remainder of their cautiously worded remarks, I believe Messrs Spottiswoode and Moulton would regard the function ascribed to the molecular streams by this "attractive hypothesis" to be that of carriers of a convective discharge out into the gas. A slight modification would however adapt the hypothesis to the very suggestive views of Messrs E. Wiedemann and H. Ebert[5].

[1] *Wied. Ann.* 11, 1880, p. 836, and *Ann.* 12, 1881, p. 276.
[2] Wiener, *Sitzungsberichte*, Bd. LXXXI., Abth. II., 1880, pp. 864—923, see specially p. 873.
[3] *Journal of Electrical Engineers*, Vol. xx., Feb. 1891, pp. 29—36.
[4] *Phil. Trans.* 1880, pp. 582 and 649—650.
[5] *Wied. Ann.* 35, 1888, pp. 217 and 258—9.

These observers found that the illumination of the cathode surface by ultra-violet light, which in general has at high gaseous pressures a remarkable effect in facilitating the discharge, ceases to have any certain effect at low pressures when the presence of phosphorescence shows the existence of molecular streams.

Their explanation is that ultra-violet light promotes the discharge by setting up at the cathode surface, apparently in the ether, vibrations of great rapidity such as would be produced by heating the cathode to a high temperature—in general a most efficacious way of promoting discharge—, but that the molecular streams are the consequence or the concomitant of the production of such rapid vibrations by some independent cause which they do not specify. Thus the function which the ultra-violet light performs at high pressures, is already fully provided for at low pressures.

The point I more specially wish to bring out is this. At high pressures whatever tends to increase the violence of the negative discharge—e.g. an air spark in the circuit on the cathode side of the tube—tends to set up the molecular streams. There seems, however, strong grounds for believing that the production of these streams actually facilitates the discharge, rendering it less violent and disruptive than it otherwise would be, especially at very low pressures.

As regards the nature of the discharge itself various views are entertained, to some of which reference is necessary to explain what follows. In the opinion of Professors J. J. Thomson[1], Schuster[2] and others the ordinary vacuum tube discharge is in a way electrolytic. The molecules of gas become dissociated and the atoms act as carriers of positive and negative electricity. The heating of the cathode or anode, as the case may be, or the presence of a heated wire, influences which have been shown by Hittorf[3], Elster and Geitel[4] and others to promote discharge in a wonderful way, operate on this theory by dissociating the gas and so furnishing atoms ready to respond at once to the directive action of the external source of electricity. At a luminous part of the discharge there is a production of heat owing to the coming together of oppositely charged atoms, at a dark part such coalitions are rare. Other writers, e.g. Lehmann[5] and E. Wiedemann[6], regard the discharge as usually of two co-

[1] *Phil. Mag.* Vol. xv., 1883, pp. 427—434; Aug. 1890, pp. 129—140; March 1891, pp. 149—171, etc.

[2] *Proceedings of the Royal Society*, Vol. xxxvii., 1884, pp. 317—339, and Vol. xlii., 1887, pp. 371—9.

[3] *Wied. Ann.* 21, 1884, pp. 106—139.

[4] *Wied. Ann.* 37, 1889, pp. 315—329, and *Ann.* 38, 1889, pp. 27—39, etc.

[5] *Molekularphysik*, Bd. ii. pp. 220 *et seq.*

[6] *Wied. Ann.* 35, 1888, p. 256.

existent but more or less independent parts, a dark probably convective discharge independent of chemical action, and a luminous discharge. Lehmann's idea is something of the following character. The luminous discharge is essentially disruptive and intermittent whatever be the nature of the source of electricity. Hittorf[1] and Homén[2] it is true, employing for the source a large number of cells, have imagined they got a steady luminous discharge through the tube like the current in a metallic conductor. But their reasons for this view such as the silence of a telephone in the circuit are, according to Lehmann[3], not conclusive, because the steadiness of the current outside the tube does not necessarily prevent its being intermittent inside. He regards the electrodes as charging and discharging like condensers, requiring a certain potential depending on the density, temperature, etc. of the gas before the luminous discharge is possible. Simultaneously, however, the electrodes leak into the tube, the discharge being carried off probably convectively without any necessary luminosity. An increased brilliancy in the tube only implies an increased quantity of electricity passing at each individual luminous discharge, and thus it accompanies whatever raises the capacity of the electrodes or affords an obstacle to rapid discharge. This explains the action of an air spark in the circuit outside the tube. A diminished brilliancy may mean the passage of a smaller current, or it may indicate an increased rapidity in the succession of luminous discharges without any alteration in the current outside the tube, or it may indicate the operation of some agency facilitating the convective discharge. It is to one or both of the latter causes that Lehmann would ascribe the effects of heating a cathode or exposing it to ultra-violet light. It is unquestionable that in many such cases the diminution in the luminosity of the tube is very striking. Lehmann[4] does not regard the luminous discharges at the anode and cathode as having any necessary relation in the rapidity of their succession. He regards a red colour as merely indicating a strong, a blue colour a weak discharge. Thus if, as usual, the positive discharge is red and the negative blue, the difference is to be accounted for either by the generally smaller cross-section of the positive column, or by the interval between successive discharges being longer at the anode than at the cathode.

A good many writers while recognizing a convective discharge ascribe it to the action of dust particles[5]. These may exist in

[1] *Wied. Ann.* 20, 1883, pp. 705—712, etc.
[2] *Wied. Ann.* 38, 1889, pp. 172 *et seq.*
[3] *Molekularphysik*, Bd. II. pp. 234—7.
[4] *Molekularphysik*, Bd. II. p. 257.
[5] See Lenard and Wolf, *Wied. Ann.* 37, 1889, pp. 443—456.

the original gas or be derived from the cathode by the disintegration which has been shown to accompany the passage of the discharge, and in many cases at least to follow the incidence of ultra-violet light.

Whatever the nature of the discharge may be it certainly does not follow Ohm's law. Hittorf and Homén, using a current which outside the vacuum tube seemed steady and continuous, found under certain limitations—depending on the spread of the negative glow over the cathode—that the potential difference at low pressures between the electrodes was nearly independent of the strength of the current. At very low pressures the fall of potential took place in great measure quite close to the cathode surface. Hittorf[1] found in the positive part of the discharge a more or less regular fall of potential, and this fall per unit length of tube was much greater than that in the non-luminous Faraday space. Thus here at least non-luminosity seems to indicate diminished resistance.

As one of the liquids I tried was mercury some points connected with discharge through Hg. vapour claim special notice. Mercury vapour being usually considered mon-atomic, it is clear that discharge through it presents some peculiarities on the electrolytic theory of discharge, there being no very obvious means of separating the gas into carriers of positive and negative electricity.

This peculiarity drew to it the attention of Professor Schuster[2] who states as the result of very careful experiments:—"I find that if the mercury vapour is sufficiently free from air, the discharge through it shows no negative glow, no dark spaces, and no stratifications." He also found the discharge to present almost exactly the same features at both electrodes. The introduction of a very slight trace of air set up the Crookes' dark space at once. Experiments in agreement with Schuster's are described by Natterer[3]. Professor Crookes[4] has, however, arrived at results diametrically opposed to those of Professor Schuster. After taking the greatest pains to prevent the presence of any foreign gas, he found a distinct Crookes' space and at least traces of a dark Faraday space. He employed aluminium electrodes, while Professor Schuster preferred them of platinum with only a small surface exposed.

Elster and Geitel[5] examining the effect of the presence of a heated wire in a tube on the contained gas, found mercury

[1] *Wied. Ann.* 20, 1883, pp. 726 *et seq.*
[2] *Proceedings of the Royal Society*, Vol. xxxvii., 1884, p. 319.
[3] *Wied. Ann.* 38, 1889, p. 669.
[4] *Journal of Electrical Engineers*, Vol. xx., Feb. 1891, pp. 44—6.
[5] *Wied. Ann.* 37, 1889, pp. 319 and 327.

vapour to possess the rare, if not unique, property of showing no electrification. This of course fits in extremely well with the electrolytic theory, which these writers seem to favour.

Gassiot[1] more than thirty years ago made some interesting experiments on mercury. His apparatus was so constructed that he could have for his electrodes either metal wires or the Hg. surface itself. Describing the appearance of the discharge as the mercury rose in the tube, he says, "as soon as the mercury ascends above the negative wire, a beautiful lambent bluish white vapour appears to arise, while a deep red stratum becomes visible on the surface of the mercury," p. 4. This red glow was only sometimes apparent. In one place Gassiot suggests it may be due to impurities in the mercury, in another he considers it analogous to the glow on a platinum wire electrode, but he seems to have arrived at no final conclusion. A mercury surface, he says, when cathode is all covered with a luminous white glow, but when anode only the extreme point is luminous. He records numerous observations on striae in tubes in which mercury was present, but it is not always easy to follow the exact conditions of the experiment. He mentions that by reducing the temperature of a vacuum tube containing mercury to $-102°$ F. or raising it to $600°$ F. he caused the striae to disappear. He also on several occasions got rid of striae at ordinary temperatures by using for cathode the surface of the mercury itself. Thus on p. 7, *Phil. Trans.* 1858, he says, "...immediately it (the mercury) covers the negative wire the stratifications disappear, and the interior of the globe is filled with bluish light." I think one may fairly conclude from his experiments that the nature of the electrodes and especially of the cathode exercises an important influence on the phenomena observed in the discharge through mercury vapour.

In the experiments now to be described I used the secondary current from an induction coil, varying the primary current according to circumstances. When the resistance in the vacuum tube circuit is sufficiently large only the direct extra current passes. The appearance of the discharge shows at once whether this is the case.

The tubes which I employed were constructed by Professor Thomson's assistant, Mr Everett, who rendered me valuable assistance in the course of the experiments. As the first tube had only a brief existence it will suffice to describe the second which resembled it in all essential points. The diagrammatic sketch, fig. 1, p. 231, will explain the general character. The nature of the electrodes varied but their position in the tube was fairly

[1] *Phil. Trans.* 1858, pp. 1—16, and 1859, pp. 137—160.

constant. For one of the vertical tubes, say *AE*, the distance of *D* above the electrode was 170 mm. of which the lowest 100 mm. was of uniform diameter, 13·5 mm. externally. Above this the tube narrowed for some 12 mm. and then continued of

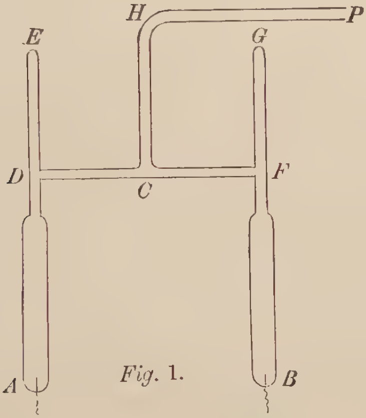


Fig. 1.

uniform external diameter 8 mm. up to *E* at 250 mm. above the electrode. The horizontal tube *DF* had an external diameter of 6 mm. and a total length of 160 mm. Thus the total distance between the electrodes was about 50 cm. These measurements were not made with any great exactness and the surfaces of the electrodes were not kept at a perfectly constant level. The tube *CHP* led to the mercury pump, *CH* being vertical, and *HP* parallel to *DF*. This tube was of about the same diameter as *DE*.

Supposing the tube *AD* to have a liquid electrode, this was connected with the exterior by a fine wire whose top was about 1 cm. below the liquid surface, and which passed down air-tight through the bottom of the tube.

The point to which Professor Thomson originally directed my attention was the question whether there was any change introduced in the liquid surface by the discharge which altered the subsequent character of the luminous appearances. As my results on this point are intelligible only when the form of the discharge has been described, their discussion is postponed to a later part of the paper.

There were three sets of experiments on mercury. In the first both electrodes were mercury surfaces; in the second one was mercury, and the other an uncovered platinum wire extending some 2 cm. up the axis of one vertical tube; in the third set the platinum wire was replaced by a flat horizontal

plate of aluminium. The gas originally in the tubes was always air, and a mercury pump was used.

In all three cases the following phenomena were observed in the Hg. tube—i.e. the branch containing a mercury electrode. At fairly high pressures when the spark first passed freely the positive discharge was bright red, as is usual in air. As the pressure diminished the colour became whiter, and there appeared numerous striae, a distinct Faraday space, and a whiteish blue negative glow. Before the striae became conspicuous the discharge left an Hg. anode from its extreme summit, but as the exhaustion proceeded it left from an increasing area, till finally the whole surface became luminous.

As the pressure was diminished there appeared what seemed to be a Crookes' space over an Hg. cathode. It was not so clearly defined as that space usually is, but the surface separating it from the negative glow was at first tolerably distinct. This surface was convex like the Hg. surface itself and had a very similar curvature. This Crookes' space was, however, by no means very dark, being in general of a distinctly red aspect, and on occasions almost fiery in appearance. The introduction of a minute air bubble into the tube made the boundary of this space appear much more distinct. As the pressure was further reduced the Crookes' space increased in length and the top of the Faraday space moved up the tube. This went on until the Faraday space extended to 80 or 90 mm. from an Hg. cathode. At this stage the stratification in the tube containing an Hg. anode was fairly distinct, the length of a stria being about 1 cm. and its luminous portion being nearly pure white. After this stage there appeared a change in the character of the phenomena most conveniently dealt with in the separate discussion of the different sets of electrodes.

Electrodes both Hg. *surfaces.*

When the stage just referred to was reached in this case, the variety in the appearances at different parts of the discharge tended to disappear, the whole assuming a more or less uniform white colour. The Faraday space seemed on some occasions certainly absent, and the Crookes' space became, to say the least of it, exceedingly indistinct. The difficulty of coming to a decision as to the existence of these spaces was much increased by the deposition of matter on the walls of the tube containing the cathode. The nature of this deposit was the same as that described in the next case. All I can say with certainty is, that the red-black space—assumed to represent Crookes' space—did not continue to expand continually as the

exhaustion proceeded. It seemed when longest to reach as far as 1 cm. above the cathode, becoming less distinct as it expanded. At the lowest pressures reached, however, while the 4 or 5 mm. of the tube immediately above the cathode were perhaps somewhat darker than the average, the luminosity had certainly attained a maximum within not more than 7 or 8 mm. of the Hg. surface. The stage at which the Faraday space became indistinct depended to some extent on the nature of the break in the induction coil. With a rapid break and quiet spark the luminosity fell off and the Faraday space remained longer distinct.

Electrodes Hg. *surface and* Pt. *wire.*

With electrodes so different in form the variations in the phenomena cannot be entirely ascribed to difference of material. According to Goldstein[1] an alteration in the size of the anode does not affect the striae in a stratified discharge, but a diminution in the size of the cathode while leaving unaltered the length of the individual striae increases the distance of the head of the positive discharge from the cathode. E. Wiedemann[2] found that the relative facility of discharge between points and between plates changes with the pressure. Thus at pressures over 1 mm. he found the discharge to pass between points and not between plates at the same distance apart, whereas at lower pressures it passed most easily between the plates. Lehmann[3] lays down some apparently general laws as to the effects of making blunter a cathode, but he does not always seem consistent on this point, and I have some doubts as to how far he bases his views on experiment. So many secondary influences are at work, such as the size of the tube and the distance of the electrodes, especially when comparable with their transverse dimensions or with the length of Crookes' space, that one would hardly expect *a priori* any simple general law to apply.

At the lower pressures the platinum wire rapidly became red hot and the deposit on the tube around it became very thick, so that it was impossible to see anything of the Crookes' space. If it did exist it must have been considerably less than the Crookes' space in the Hg. tube when that was last seen distinctly. The comparison instituted was thus between the distances from the cathodes of the further extremity of the Faraday space, i.e. the head of the positive discharge. The following are some of the data obtained, the simultaneous lengths of the Crookes' space at the Hg. cathode being given when noted:

1 *Wied. Ann.* 12, 1881, p. 275.
2 *Wied. Ann.* 20, 1883, pp. 795—7.
3 *Molekularphysik*, Bd. II. pp. 277—8, and *Wied. Ann.* 22, 1884, pp. 320—1.

TABLE III.

Hg. Cathode		Pt. Cathode
Length of Crookes' space.	Distance above cathode of head of positive discharge.	Distance above cathode of head of positive discharge.
	30	13
	35	14
4	60	20
5	65	20
7	75	20

The distances are in mm., and are all measured from the upper extremity of the cathode. The observations recorded in the same horizontal line were taken in immediate succession, the make and break regulator of the induction coil remaining untouched. The uniformity of the pressure and of the make and break was tested by reversing the current twice, which showed whether the appearances at the surface which was first cathode had altered during the observations. The experiments extended over a considerable interval and the tubes were cleaned out and refilled more than once, thus the difference between the cathodes shown by the table, which was confirmed by numerous observations in which accurate measurements were not taken may, I think, be fully relied on. The table shows that the distance above the cathode of the head of the positive column was on an average about thrice as much when the cathode was mercury as when it was platinum. The difference was greater the lower the pressure. At the highest pressures when the Faraday space first became distinct no exact measurements were recorded, but the difference though then not so striking as in the table was still conspicuous. As the exhaustion proceeded very slowly and both tubes were at intervals heated up by a burner, the difference can hardly be attributed to any great extent to a difference in the gaseous contents of the two vertical tubes.

The difference may be due to the difference in the material of the electrodes or to the dissimilarity of their sizes and shapes. According to Goldstein the effect of the difference in size of the cathodes should have tended in the opposite direction. For reasons explained in treating of the next case, I am inclined to attribute the difference in considerable measure to the difference in shape.

Appearance of Discharge.

The tube containing the Pt. electrode always showed a redder tint than that containing the Hg. electrode under similar conditions.

At a certain stage of exhaustion the former tube was sometimes the redder even when the Hg. surface was the anode. At the lowest pressures, however, the difference between the colour of the tubes was inconspicuous, both being prevailingly white. At the highest pressures the spark left the tip of the Pt. anode and the extreme summit of the Hg. anode, but as the exhaustion proceeded it gradually extended down the Pt. anode and spread over the surface of the Hg. anode. Eventually the positive column covered the whole of the Hg. surface, but within 1 or 2 mm. up the tube it had contracted somewhat in diameter, leaving an annular dark space between it and the glass. The Faraday space became gradually indistinct over the Hg. cathode, and the same phenomenon appeared over the Pt. cathode but at a lower pressure. Thus at one stage of the exhaustion the appearances in the two tubes when containing the cathode were widely different. At the lowest pressures reached, both tubes, so far as clearly seen, were very similar in appearance, and the phenomena agreed with those observed when both electrodes were of mercury.

Phosphorescence.

For clearness let us suppose, as was actually the case, A the Hg., B the Pt. electrode. In the tube BG the phosphorescence first appeared at the level of the upper portion of the wire, and gradually spread both up and down as the exhaustion proceeded, reaching the summit G of that tube. In the tube AE the phosphorescence hardly appeared within 2 cm. of the Hg. surface. In the lower parts of both tubes phosphorescence was most brilliant at about the positions of the Faraday spaces at the lowest pressures, and so at a much higher level in AE than in BG. Also the phosphorescence at E was much more intense than that at G, though the latter was very well marked.

At the lowest pressures reached a faint nebulosity, presumably the positive column, extended to a considerable distance along the tube CHP. It was difficult to detect when the platinum was cathode, but when the mercury was cathode it could be traced almost as far as the pump.

There then appeared at H a patch of phosphorescence on the convex side of the bend. This was very faint when the platinum was cathode, but when the mercury was cathode it was fairly

bright. This is clearly a variety of the phenomena observed by Goldstein and by Spottiswoode and Moulton, but it seems worth noticing as the tube CHP did not lead to an anode. The Hg. column in the pump did not lead to earth, and further the luminosity in HP decreased as the distance from H increased, which it would hardly have done if the pump had acted as anode.

Deposit on the tubes.

In the tube BG containing the platinum there were two, generally distinct, principal areas of deposit on the glass. When the platinum had served for some time as cathode there was a dense black deposit from a little above the level of the top of the wire downwards, and a second less dense deposit separated in general from the lower by an almost perfectly clean and sharply defined area only about 1 mm. broad but extending right round the tube. Roughly speaking, the upper deposit extended to about the highest level attained by the extremity of the Faraday space, but there were traces of it further up the tube. Thus the portions of the walls of this tube where the deposit was thickest were precisely those where the phosphorescence was strongest ere the darkening of the glass reduced its brightness. At the same time phosphorescence was conspicuous on the narrow patch of clean glass separating the two principal areas of deposit, and also in the upper portion of the tube FG where no deposit was seen.

In the tube AE the *permanent* deposit extended in general from 20 or 30 to 80 or 90 mm. above the Hg. surface, none appearing in the neighbourhood of E. It was nowhere so thick as that on the other tube. Looking at it in strong light one could see small drops of mercury scattered about, but its exact composition was not determined. There were other deposits of a more temporary character near the Hg. surface. One had the appearance of dew spreading up the tube for a few millimetres when the Hg. was made cathode, and gradually creeping down when the current was stopped, taking only a short time to disappear. It was seen only at low pressures. In some cases there was a sort of white deposit separated from the Hg. surface by a clear space of a few millimetres, which increased apparently as the exhaustion proceeded. On readmitting air to the tube this last form of deposit in great measure disappeared. What has been called above the permanent deposit had a blackish colour and was but little affected by the readmission of air. These deposits by obscuring the portions of the tube where the Faraday and Crookes' spaces had to be looked for, increased very much the difficulty of reaching final conclusions as to the existence of these spaces at the lowest pressures.

Electrodes Hg. *and* Al.

The aluminium electrode was a flat circular plate of about two-thirds the internal diameter of the tube. The following observations were taken on several occasions:

TABLE IV.

Hg. Cathode			Al. Cathode		
Length of Crookes' space.	Distance above cathode of top of negative glow.	Distance above cathode of head of positive discharge.	Length of Crookes' space.	Distance above cathode of top of negative glow.	Distance above cathode of head of positive discharge.
·5		18			11
	8	20		5	12
·5	11	22		7	13
1	15	23		12·5	16·5
·75		24			18
		25			17
		30			20
1·5		40			32·5
2		40			30
		40			25
8		80	6		70

The distances are in millimetres, and were all measured from the centre of the cathode surface. At the higher pressures the upper limit of the negative glow was pretty distinct and so has been recorded above. Observations in the same horizontal line were taken in rapid succession as in the case of Table III. The ratio of the mean distance from the cathode to the end of the positive column when the cathode was mercury to the corresponding mean distance when the cathode was aluminium is as 1·37 : 1, and so is very much less than the ratio found when the solid electrode was platinum. Since no such striking difference between the metals platinum and aluminium as electrodes seems to have been noticed by previous observers, the inference would seem to be that we are here concerned with the size and shape rather than with the material of the cathode.

Owing to the appearance of a slight crack in the tube with the Al. electrode its base had to be immersed in mercury contained in a paraffin cup. Thus the length of the Crookes' space when short could not be accurately observed without lowering the cup. This was not done in the cases recorded in the table, to avoid the risk of leakage, variation of current, etc., but on

other occasions such observations were taken and it was found that from the highest pressures where it was visible down to the lowest in the above table, Crookes' space appeared slightly but distinctly longer over an Hg. cathode than over an Al. cathode.

Appearance of the discharge.

Under similar conditions the Al. tube was always redder than the Hg. tube, though both were at the lowest pressures mainly white. When the pressure was reduced to a certain stage, the Faraday space over the Hg. cathode became more and more indistinct till it seemed to vanish. The Faraday space over the Al. cathode was at this stage unmistakeable, but at a lower pressure it too eventually became undistinguishable. The way in which the Faraday space over the Hg. cathode disappeared was rather remarkable.

The pressure reached a point at which the appearance in the tube was unstable. There might be an unmistakeable Faraday space and negative glow, and then a sudden transition to a stage in which distinct striae reached down the tube to near the Hg. surface. During this time the discharge in the Al. tube showed no fluctuation. As the pressure was carried lower the striae became less and less distinct until the Hg. tube whether the Hg. were cathode or anode seemed an almost uniform white. The lowest 5 or 6 mm. in the tube appeared sometimes perceptibly, sometimes doubtfully darker than the rest. Distinct striae eventually ceased to appear even in the Al. tube, but it showed to the end an unmistakeable Crookes' space which however was becoming increasingly indistinct at the lowest pressures reached.

The greatest length reached by the Crookes' space over the Al. cathode was 14 mm. At this stage the tube containing the Hg. cathode was as bright as anywhere within 7 or 8 mm. of the mercury surface.

Phosphorescence.

When the aluminium was cathode phosphorescence extended at the lowest pressure throughout the whole of the tube BG, whereas when the mercury was cathode phosphorescence was not observed within some 2 cm. of its surface. The phosphorescence at the top E of the Hg. tube was always brighter than that at G at the same exhaustion. At the best exhaustion with the Hg. cathode there was distinct luminosity along the tube CHP as far as the pump, a distance of 42 cm. from H, and a patch of phosphorescence was distinctly visible at H. At the

same exhaustion with the Al. cathode a faint luminosity was seen to some distance past H, and a very faint phosphorescence could be made out at the bend.

Deposit on the tubes.

In the Hg. tube there were the same deposits as before, and a further one was observed under the following circumstances.

On several occasions on examining the Hg. surface by daylight it was found to present a yellow metallic appearance. The conditions preceding its first appearance were as follows. An air-bubble which had remained under the mercury was driven up whilst the discharge was passing. Mercury splashed up the walls of the tube and adhering to some extent presented a concave surface. Some observations were taken with the surface in this state, and next day it was noticed that the surface was yellowish, and that there were traces of a yellow deposit not only up the tube AE but also at intervals along DF and even for some distance down FB.

On another occasion when the tube had just been carefully cleaned and dried and fresh mercury introduced, I noticed after passing the spark both ways for some time that the Hg. surface though retaining its ordinary convexity had a decidedly yellow appearance, and that the tube near it was slightly yellow. Air was allowed to leak in, and the tube being left for some days appeared when next examined quite clean, while the Hg. surface had its usual colour. When however the tube was again exhausted and left for some days, the Hg. surface and a few millimetres at the base of the tube were found yellow as before. However, on passing the spark the phenomena had their normal character—which was not the case when the mercury surface was concave—and on re-examining the tube the yellow colour was found to have entirely disappeared and it was not observed again. With the exception of the yellow patches above mentioned and a narrow dark ring sometimes observed on the glass near the head of the positive column, the tube with the Al. electrode showed no distinct deposit. The origin of the yellow colour was not discovered. Its appearance in the tube FB suggests but does not prove a capacity in the discharge to transport particles from a cathode surface round corners. The transporting agency might of course have been vapour rising from the Hg. surface and condensing on cooler portions of the tube.

Effect of sudden alteration of the Hg. *surface.*

The experiments on this point were those first carried out. Both electrodes were then of mercury, and the tube CHP instead

of being fused to the pump as it was during the other experiments, was connected to it by some thick-walled india-rubber tubing. The mode of altering the surface was simply by shaking or vigorously tapping the tube. With the head of the positive column from 9 to 30 mm. above the cathode, the stage at which the Faraday space was most distinct, no certain change in its position was observed to follow the alteration of the surface.

The only stage of exhaustion at which a distinct effect of any kind was observed was that where the Faraday space seems to be in the act of disappearing. On first starting the current, especially with a slow break and noisy spark, there appeared more or less uniform whiteness in the cathode tube and indistinct striae in the anode tube. After the discharge had passed a short time, the colour tended to fade out of a portion of the cathode tube, roughly speaking, between 30 and 90 mm. over the Hg. surface, and phosphorescence became much more conspicuous, especially in this part of the tube; also the striae in the anode tube became more distinct. A shaking or sharp tapping of the tube instantly restored the more or less uniform white colour throughout the cathode tube and tended to obliterate the striae in the other tube. The effect lasted only a short time, the discharge gradually reverting to the appearance it presented before the disturbance. This phenomenon invariably presented itself under the conditions stated. The only explanation that occurs to me—suggested by the views of Messrs E. Wiedemann and H. Ebert—is that, at least at certain stages of exhaustion, the condition at the cathode surface which leads to the projection of molecular streams takes some time for its full development, and that on its development the successive discharges follow one another more rapidly and consist each of a smaller quantity of electricity. The shaking of the tube and consequent distribution of fresh mercury over the cathode surface restores the original conditions, which are less favourable to the production of molecular streams.

Electrodes H_2SO_4 *and* Al.

In the next set of experiments the electrode in AE was some pure sulphuric acid, the aluminium plate electrode in the other tube being retained.

Some experiments with sulphuric acid electrodes have been described by Paalzow[1]. He gives an interesting account of the electrolysis of the acid, and of spectroscopic observations on the discharge. He observed the positive discharge to start from the line of separation of the fluid surface and the wall of the tube.

[1] *Wied. Ann.* 7, 1879, pp. 130—135.

As I hardly follow his description of the appearances at the cathode I give his own words: "Von der negativen Flüssigkeitsoberfläche selbst erhebt sich in einigem Abstande von derselben ein schwach conischer Lichtring, ähnlich wie die Flamme eines ringförmigen Brenners," p. 131. With increasing exhaustion: "um so mehr verlängert sich dieser negative Lichtcylinder, und um so grösser wird sein Abstand von der Flüssigkeitsoberfläche," p. 132. At very low pressures he observed the phenomena to be much the same at both electrodes.

In this case my observations commenced as soon as the pressure was sufficiently reduced for the discharge to become visible in a dim light. This invariably occurred when the aluminium was cathode, and the luminosity took the form of a thin purplish negative glow. At somewhat lower pressures the Al. tube was fairly luminous whether it contained the cathode or anode, the H_2SO_4 surface when cathode showing a thin blue glow. The horizontal tube and a small portion of the tube *AE* below *D* then showed a red spark discharge. The greater portion of the latter tube remained however dark, except that at intervals red twig-like discharges passed down it. At this stage the spark in the tube *DF* was most twig-like and of least diameter at that end which was nearest the cathode, whether Al. or H_2SO_4. The pressure had to be further reduced to a considerable extent before luminosity could be detected at the H_2SO_4 surface when anode. When the discharge was first clearly seen at an H_2SO_4 anode it took the form observed at pretty low pressures with an Hg. anode. Throughout the greater portion of the tube the positive column appeared as a solid cylinder of considerably less diameter than the interior of the tube, but near its base this cylinder increased in diameter so as just to fill the tube on reaching the liquid surface.

At this stage the appearance in the tube *AE* over an H_2SO_4 cathode took the following form. A column of very small diameter, usually bright red in colour, extended down the axis of the tube to within a short distance of the liquid surface. The end of this column was sometimes truncated, but frequently it presented a sharp point like that of a pencil. Over the liquid surface, completely occupying the cross-section of the tube, there existed the ordinary blue negative glow. At first this was separated by several millimetres of a dark, presumably Faraday, space from the red column, but as the exhaustion proceeded while the head of the column retired from the H_2SO_4 surface the negative glow overtook it. With progressing exhaustion the point and a gradually increasing length of the red column were immersed in the blue glow. The difference in colour rendered the red column conspicuous through the blue glow, but I am unable to say

whether the two were actually in contact. Possibly they may have been separated by a thin hollow conical dark space.

At this stage the tube containing an Al. cathode showed a positive column whose diameter was several times greater than that of the column just described. Further, the head of the positive column over an Al. cathode had the usual convexity, and was separated by a distinct Faraday space from the negative glow. As the exhaustion proceeded an ordinary Crookes' space appeared over an Al. cathode, and the other phenomena seemed of the usual type. Over an H_2SO_4 cathode, however, the phenomena seemed of an exceptional kind.

The liquid surface was of course concave, the depression of the vertex below the rim being about $1\frac{3}{4}$ mm. Thus on looking sideways at the tube one saw a curved dark line answering to a vertical section of the liquid surface. Between this and the horizontal plane through the rim of the surface there appeared a blue tint. This might have proceeded from a luminous film over the liquid surface or merely from the reflexion of the glow further up the tube. Now supposing a Crookes' space to exist, the surface separating it from the negative glow would by analogy from other cases be expected to resemble in form the liquid surface. Thus so long as the thickness of this space was less than the sagitta of the liquid meniscus, one would have expected the blue negative glow to pass near the axis of the tube into the blue of the liquid surface, while close to each of the tube walls one would look for a dark space shaped like a right-angled triangle, with a curved hypothenuse answering to the surface of the negative glow. Let us call this type (a). An idea of

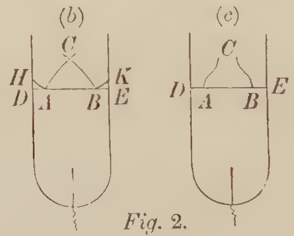


Fig. 2.

it will be easily derived from (b), fig. 2, by supposing ACB non-existent. The spaces HAD and KBE are to be supposed black, while below $DABE$ and above $HABK$ blue prevails for some distance. $DABE$ represents the liquid rim cutting the surface $HABK$ of the negative glow in the straight line AB. At lower pressures one would have expected to see the length AB gradually diminish, till finally there appeared over the whole

liquid surface a black space, which would appear to the eye to be bounded below by a straight line and above by the curve of intersection of the negative glow by a vertical plane. Let us call this type (*d*). These types were actually observed, but in addition two other types were seen, viz. those represented by (*b*) and (*c*), fig. 2.

In both, *DABE* repiesents the rim of the liquid surface as seen by an observer's eye at the same level. In (*b*) the small areas *HAD*, *KBE* had a black appearance, but were in general not conspicuous. The negative glow extended from *HABK* for some distance up the tube as in type (*a*). But it now showed a sort of tuft *ACB* of a much whiter and less translucent blue than the rest, with more or less distinct curvilinear boundaries which seemed to cross at *C* and gradually fade away beyond it. In (*c*) there was no trace of any black spaces, and the tube appeared prevailingly blue all the way up with the exception of an almost pure white tuft *ACB*. It had a tolerably distinct outline except at the top *C* where the colour passed gradually into blue. It was sometimes more conical and sometimes more depressed seemingly than in the figure, but exact measurements were not attempted.

A slight unsteadiness in the type (*c*) happened to catch the eye when looking at the liquid surface while putting on the current. Further examination led to the following results.

After keeping the pump at work for some time and then suddenly starting the discharge with the H_2SO_4 as cathode one saw a sort of white cloud instantly gather in the tube, separated apparently from the liquid surface by from 2 to 4 mm. of a dark interval. The cloud, however, immediately stretched down the tube and transformed into the appearance (*c*). This transformation was on several occasions so distinctly seen that the observer could hardly be mistaken; but it was not always seen under these conditions. The whole thing happened so fast that sometimes all one could say was that some rapid change had occurred. When the current was simply stopped and renewed without intervening exhaustion, the tuft appeared at once in the position shown in the figure.

I am not aware of previous observations on the form of the negative glow near concave cathodes forming surfaces of revolution, and it would obviously be very difficult to see what actually exists within the rim of such a cathode. Professor Crookes, however, in Plate 14, *Phil. Trans.* 1879, gives some beautiful coloured illustrations which show the negative glow and the Crookes' space near a concave cylindrical cathode. I would more especially draw attention to his illustrations *b* and *c* fig. 11, which show a great concentration of negative glow near the plane of symmetry through the axis of the cathode. In Crookes' fig. *b* there is a

regular tuft, and in his figure *c* the appearance of a bundle of rays projecting into a dark Faraday space. In fact if in these figures all to the left of the plane through the rim of the cathode were removed there would be a considerable resemblance to the phenomena illustrated by my (*b*) and (*c*) fig. 2. The principal difference is that with Crookes' electrode there appears to have been a Faraday space limiting a distinct and, except in the plane of symmetry, thin negative glow, whereas with the H_2SO_4 cathode no such distinct Faraday space was seen.

I would also call attention to Crookes' fig. 2, p. 643, *Phil. Trans.*, 1879, as showing a concentration of negative glow in positions opposite the hollows of a corrugated cathode.

The resemblance of the boundaries of *ACB* in (*b*) and (*c*) fig. 2 to caustic surfaces unquestionably suggests that we may have here to do with some species of emission from the surface separating the liquid and gas, each element of the surface acting as a source more or less independent of its neighbours. If, as seems to be the case with the molecular streams[1], the direction of emission from the elements near the rim deviates more than elsewhere from the normal to the surface, one can easily see that there would be a crossing of the trajectories near the axis of the tube, even close to the liquid surface.

The sudden change noticed when the type (*c*) fig. 2 appeared, might be accounted for by the discharge leaving at first from only a portion of the liquid surface, or possibly it may be connected with the rise of the small bubbles accompanying the electrolysis which take some short time to reach the surface.

The following table gives results from several series of observations, all data in a horizontal line being taken in immediate succession with a constant rate of make and break. All the distances are in millimetres. Except at the highest pressures even the approximate position of the upper end of the negative glow could not be fixed. Such an entry as "7 +" in the column headed "Distance...glow" means that the glow reached to some undetermined height above the point of the red column. In the case of the H_2SO_4 cathode all the distances were measured from the rim as the most convenient starting-point. It ought to be remembered, however, in instituting any comparisons that in the axis of the tube the true liquid surface was some $1\frac{3}{4}$ mm. below the rim. The negative glow in type (*d*) had a considerably greater curvature than the liquid surface, so that its distance from that surface, measured parallel to the axis of the tube, was least in the axis.

[1] See Goldstein, *Wied. Ann.* 15, 1882, pp. 254—277, specially pp. 274—5.

TABLE V.

H_2SO_4 cathode				Al. cathode		
Crookes' space: Type	Crookes' space: Height above rim.	Distance above rim of top of negative glow.	Distance above rim of head of positive discharge.	Length of Crookes' space.	Distance above cathode of top of negative glow.	Distance above cathode of head of positive discharge.
		4	5			11
		6	6			17
		7 +	7		9	19
		7 +	7			22
		13	10		17	29
(*a*)		10 +	10	2·5		
		13 +	13	4		40
		15 +	15		23	38
(*b*)			absent	4·5		
(*c*)			absent	5		
(*d*)	1·5	17 +	17	5		dubious
(*d*)	1·5	18 +	18	7		50
(*d*)	2	24 +	24	8		dubious
(*d*)	2·5	30 +	30	8		dubious

None of the distances admitted of very accurate observation because the positions of the various parts of the discharge, especially the head of the positive column over an Al. cathode, were seldom quite stationary. At the lower pressures in the table the blue light surrounding the red column over an H_2SO_4 cathode extended up the tube AE to above the level of the horizontal tube. When, however, the interrupter was adjusted to give a slower and more noisy spark the red column sometimes completely disappeared. It also sometimes faded out when the discharge was kept passing for some time. In either case there appeared from about 20 to 40 mm. above the H_2SO_4 cathode a considerably darker blue than elsewhere verging towards black. This darker band always accompanied the types (*b*) and (*c*) of discharge with which the red column was not observed. Once or twice this column instead of being red was white. In general its outline appeared straight and regular, but on at least one occasion it showed numerous short horizontal projections like hairs.

The colour of the positive column at the higher pressures and until the Crookes' space over the Al. cathode attained a length of 1 or 2 mm. was always red. At lower pressures when red its colour was much fainter, and at the lowest pressures it generally tended to white or even blue. The striae were most conspicuous when the colour was white, but they were seldom very distinct. At the lowest pressures only faint phosphorescence was observed in

the tube over an Al. cathode, and none was noticed over an H_2SO_4 cathode. There was also no deposit on the walls of the tube. The joint absence of these two phenomena is rather striking, but the exhaustion was not, I think, carried so low as in the previous experiments.

At the lowest pressure attained, with a Crookes' space of from 9 to 10 mm. in length over an Al. cathode, the appearance of the discharge became unsettled. There was nowhere any very bright colour, but throughout the greater part of the tube over an H_2SO_4 cathode the colour along the axis was unmistakeably red. There was a red column somewhat resembling that seen at higher pressures, terminating in a sharp point about 40 mm. over the H_2SO_4 surface. Immediately below this the tube appeared blue throughout the entire cross section, but a little lower down there appeared a red column of about half the diameter of the tube. Its base was curved, and its convexity was directed towards the H_2SO_4 surface, from which it was separated by an interval of only 2 to 4 mm. The colour of this interval seemed to vary from black to faint blue. Both these red columns had a blue haze between them and the walls of the tube. This stage seems to answer to that observed with a mercury electrode previous to the discharge assuming a nearly uniform appearance throughout.

My best thanks are due to Professor Thomson for putting at my disposal the necessary apparatus, and for numerous suggestions during the course of the experiments, which were performed in the Cavendish Laboratory.

(5) *Note on a Problem in the Linear Conduction of Heat.* By G. H. Bryan, M.A., St Peter's College.

The problem of conduction of heat in a bar one end of which is subject to radiation while the other end is at an infinite distance away, has been treated by Mr Hobson in his paper "On a Radiation Problem" published in the *Proceedings*, Vol. VI., page 184. The author there finds the expressions in the form of definite integrals, representing the temperature due to the given distribution of heat in the medium at the extremity of the bar and to the initial distribution of heat in the bar respectively.

The expression which Mr Hobson obtains for the second part of the temperature is, however, open to several objections. It appears to fail entirely if the initial temperature is anywhere discontinuous or if any sources, doublets, or other singularities are initially present in the bar.

Moreover, from the integral obtained, it is shown that the initial distribution is equivalent to a certain distribution of lines of sources and sinks in a rod extending to infinity in both directions,

but this interpretation is also liable to objection. For the distribution of sources and sinks corresponding to the initial temperature in any single element of the rod is *not* in itself equivalent to that element. Hence Mr Hobson's solution gives no idea of the part played by the initial temperatures of the separate elements of the rod, and in fact it does not even give a correct result if the effect of the initial temperature in any part of the rod is considered apart from that in the rest.

The problem is best solved by starting with a single instantaneous source at one point of the rod, since from this the more general solution can be obtained by integration.

Consider then the effect of an instantaneous source Q of heat generated at the point x' at the time 0. The temperature due to such a source if there were no boundary would be

$$\frac{Q}{2\sqrt{\pi kt}} \exp - \frac{(x-x')^2}{4kt}.$$

Let this expression be denoted by v_2, and let the temperature when the boundary is taken into account be v where

$$v = v_1 + v_2.$$

If the external medium be at temperature zero, the boundary condition gives, when $x = 0$,

$$\frac{dv}{dx} - hv = 0.$$

Hence
$$\frac{dv_2}{dx} - hv = -\left(\frac{d}{dx} - h\right) \frac{Q}{2\sqrt{\pi kt}} \exp - \frac{(x-x')^2}{4kt}$$
$$= \frac{Q}{2\sqrt{\pi kt}} \left\{ -\frac{x'}{2kt} + h \right\} \exp - \frac{x'^2}{4kt} \text{ when } x = 0.$$

But v_2 must be the temperature due to a series of images on the negative side of the origin, hence the conditions of the problem will be satisfied by taking

$$\frac{dv_2}{dx} - hv_2 = \frac{Q}{2\sqrt{\pi kt}} \left\{ -\frac{x+x'}{2kt} + h \right\} \exp - \frac{(x+x')^2}{4kt}$$
$$= \left(\frac{d}{dx} + h\right) \frac{Q}{2\sqrt{\pi kt}} \exp - \frac{(x+x')^2}{4kt}$$
$$= \left(\frac{d}{dx} - h\right) \frac{Q}{2\sqrt{\pi kt}} \exp - \frac{(x+x')^2}{4kt}$$
$$+ \frac{2hQ}{2\sqrt{\pi kt}} \exp - \frac{(x+x')^2}{4kt}.$$

Therefore by integration

$$v_2 = \frac{Q}{2\sqrt{\pi kt}} \exp - \frac{(x+x')^2}{4kt} - e^{hx} \int_x^\infty \frac{2hQ}{2\sqrt{\pi kt}} e^{-h\xi} \exp - \frac{(\xi + x')^2}{4kt} d\xi.$$

In the last integral put $\xi = x + z$. Also add v_1, thus the whole temperature assumes the form

$$v = v_1 + v_2 = \frac{Q}{2\sqrt{\pi kt}} \left\{ \exp - \frac{(x-x')^2}{4kt} + \exp - \frac{(x+x')^2}{4kt} \right\}$$
$$- 2h \int_0^\infty \frac{Qe^{-hz}}{2\sqrt{\pi kt}} \exp - \frac{(x+z+x')^2}{4kt} . dz.$$

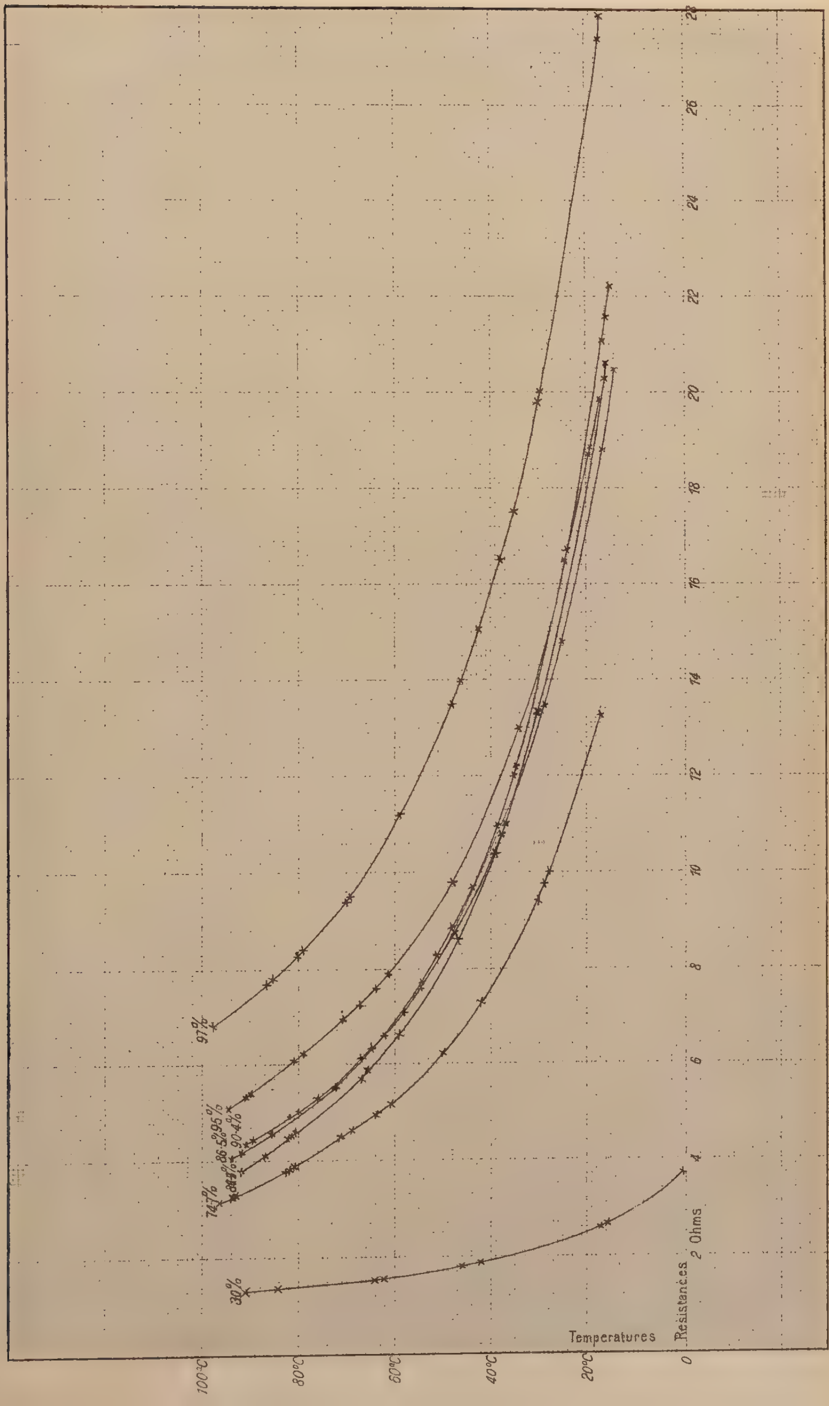
The first line represents the temperature due to the given source and an equal source at the geometrical image, the second line represents the temperature due to a line of sinks extending from the geometrical image to infinity in the negative direction. None of the images of the source lie on the positive side of the origin.

For the temperature due to the initial distribution $v = \phi(x)$ we write $\phi(x')\,dx'$ for Q, and deduce by integration,

$$v = \frac{1}{2\sqrt{\pi kt}} \int_0^\infty \left\{ \exp - \frac{(x-x')^2}{4kt} + \exp - \frac{(x+x')^2}{4kt} \right\} \phi(x')\,dx'$$
$$- \frac{2h}{2\sqrt{\pi kt}} \int_0^\infty \int_0^\infty \exp(-hz) . \exp - \frac{(x+x')^2}{4kt} \phi(x')\,dx'dz.$$

This solution holds good whether $\phi(x')$ be finite and continuous or not.

[*Note by Mr Hobson.*—Mr Bryan's formula is undoubtedly an improvement on the one I gave in the paper referred to. It should be observed that his formula may be obtained from mine by integration by parts.]


97%
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86·5%
74%
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Temperatures
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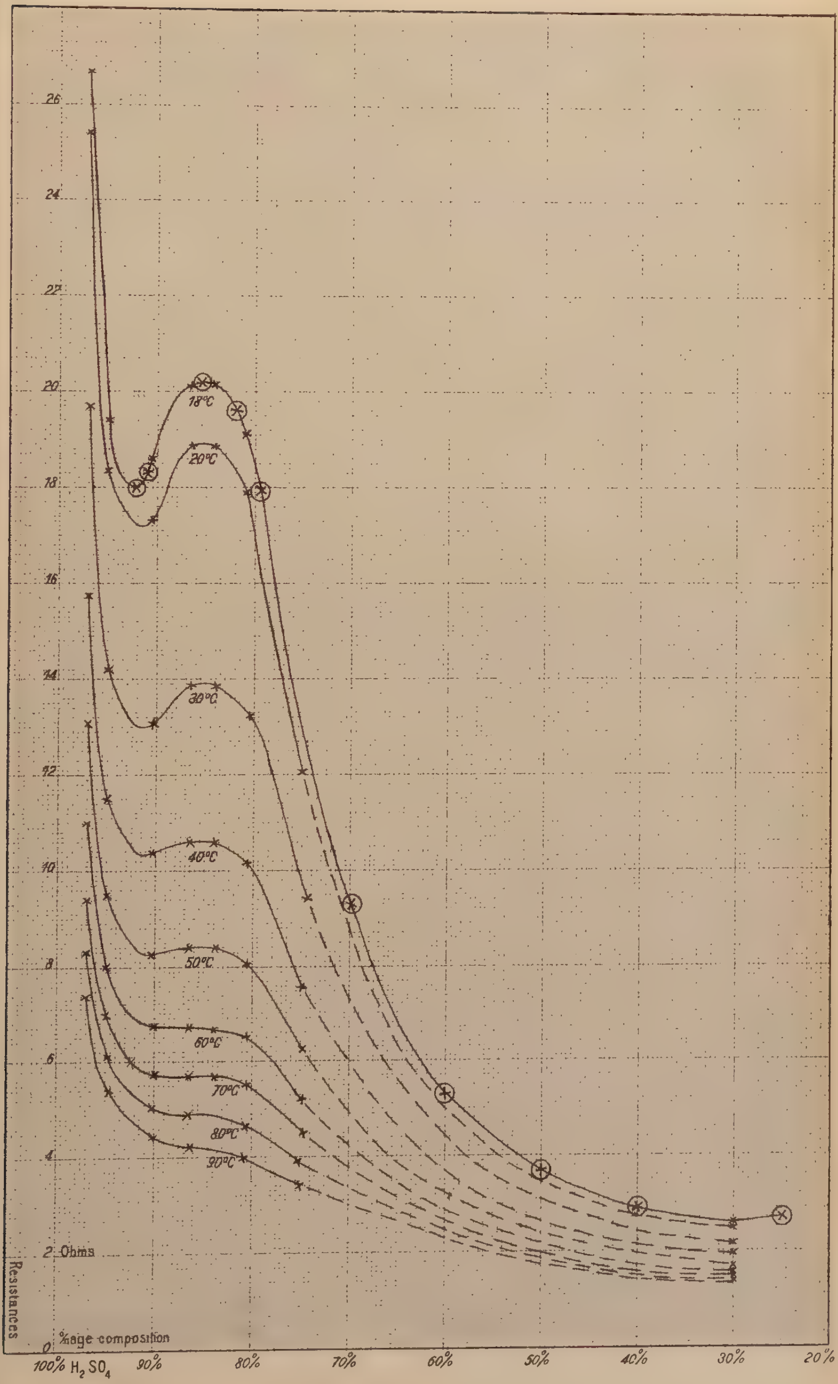

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100% H_2SO_4
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70%
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50%
40%
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PROCEEDINGS

OF THE

Cambridge Philosophical Society.

October 26, 1891.

ANNUAL GENERAL MEETING.

Professor G. H. Darwin, President, in the Chair.

The following Fellows were elected Officers and new Members of Council for the ensuing year:

President:

Prof. G. H. Darwin.

Vice-Presidents:

Prof. Hughes, Prof. Thomson, Mr J. W. Clark.

Treasurer:

Mr R. T. Glazebrook.

Secretaries:

Mr J. Larmor, Mr S. F. Harmer, Mr E. W. Hobson.

New Members of Council:

Mr H. F. Newall, Mr C. T. Heycock, Mr A. E. H. Love.

The following Communications were made to the Society:

(1) *On the Absorption of Energy by the Secondary of a Transformer.* By Prof. Thomson.

(3) *Exhibition of* Phylloxera vastatrix. By A. E. SHIPLEY, M.A., Christ's College.

PROFESSOR HUGHES IN THE CHAIR.

(4) *The Digestive Processes of* Ammocœtes. By Miss R. ALCOCK (communicated by Dr GASKELL).

[*Received December* 3, 1891.]

In all the higher vertebrates digestion is carried on by means of the secretion of specialised glands localised in certain definite portions of the alimentary canal or of glandular masses which are formed as appendages of the same. The formation of a peptic ferment is confined to the glands of the stomach, of a tryptic ferment to the pancreas and of diastatic ferments to the salivary glands and pancreas. Passing to lower forms we find in Amphibia that the formation of peptic ferment is not restricted to the stomach, but extends oralwards into the œsophagus, which is even more active as an organ for digestion than the stomach. Then in Fishes this tendency to diffuseness in the position of the proteid-digesting glands is even more pronounced, and it is remarkable, as Krukenberg has pointed out, how their position varies even in nearly allied families. Sometimes a pancreas is present, sometimes absent; in some the appendices pyloricæ are well developed, in others they do not exist, and in some they function as digestive glands, whilst frequently they seem merely to act as organs of absorption. In some cases the so-called pancreas does not function in proteid digestion, and in many fishes certain cells forming part of the liver secrete a tryptic ferment, which enters the alimentary canal by means of the bile-duct. The pepsin-forming glands also vary in position, sometimes extending oralwards and sometimes into the upper part of the intestine.

These observations of Krukenberg lead to the conclusion that, as far as digestive organs are concerned, the lower we descend in the scale of evolution of the vertebrates, the greater is the tendency towards a decrease of specialisation in function and a diffuseness in the position of the secreting cells. If this is the case, the study of the digestive processes in the lowest vertebrates ought to shew this absence of concentration of the secreting tissues in a still more pronounced manner. With this object Dr Gaskell proposed to me to find out how the digestive processes were carried on in the Ammocœtes, as no physiological observations have been made on the digestion of this primitive vertebrate by Krukenberg or any other observer.

We may consider for this purpose that the alimentary canal consists of three portions, 1st the pharynx, 2nd the narrow tube or anterior intestine which leads from the pharynx to the mid-gut and terminates posteriorly at the entrance of the bile-duct where the sudden enlargement of the alimentary tube marks the beginning of the 3rd portion, the intestine proper. The glandular appendages in connection with these parts are, (1) the so-called thyroid gland, with its duct opening into the pharynx, and (2) the liver with its duct opening into the intestine.

In order to test for the presence of digestive ferments, I made extracts of the epithelium lining the pharynx, the liver, the intestine and the thyroid in ·2 °/$_{\circ}$ HCl. or in glycerine, using in each experiment the organs of two or more Ammocœtes of *Petromyzon Planeri.* I may here mention that I can confirm Krukenberg's experiments on the digestive ferments in fishes and invertebrates as to the temperature at which they are most active. I find my extracts are very much more powerful at 38° to 40° C. than at 15° to 20° C., many authors giving the lower temperature as that at which the digestive extracts of fishes and cold-blooded animals generally are most active. I found that all parts of the alimentary canal, with the exception of the thyroid gland, were capable of digesting fibrin in a ·2 °/$_{\circ}$ HCl. medium with greater or less rapidity. I used carmine-stained fibrin after Grützner's method, and always used as control experiment a ·2 °/$_{\circ}$ HCl. solution alone and an extract of some tissue of the animal which was inactive in digestion.

The results of these experiments may be summed up as follows:—The extract of the liver is always the most powerful;—thus in one case about 1 c.c. fibrin was entirely digested in from $\frac{1}{2}$ to $\frac{3}{4}$ hr. by an extract made from the livers of two Ammocœtes. The epithelium of the pharynx comes next in activity, thus in the case mentioned the epithelium of the pharynx of the same two animals digested the same amount of fibrin in about $1\frac{1}{2}$ hrs. Next in order of activity comes the extract of the intestine; this always gives decided evidence of digestive power, but if carefully cleaned out with a soft brush before the extract is made, so as to remove as far as possible the secretion from the liver, then its power of digestion is very far behind that of the pharynx or liver, although in all cases digestive activity is still manifest. Finally the extract of the thyroid is always inactive.

So far I have not succeeded in obtaining any results in a 1 °/$_{\circ}$ sodium carbonate medium, and conclude that tryptic ferment is absent. In relation to this it is interesting that in a young Selachian Krukenberg found that a tryptic ferment was entirely wanting, and he suggests, for this and other reasons, that in primitive vertebrates the digestion was rather peptic than tryptic.

I conclude then from these experiments that:—

1. The proteid digestive ferment in the Ammocœtes is of the nature of pepsin rather than trypsin.

2. This ferment is very diffuse in position, being found in all parts of the alimentary tract.

3. It is found mainly in the anterior part of the tract, especially in the respiratory portion of the pharynx and in the liver.

4. The so-called thyroid gland has nothing whatever to do with the digestion of proteids.

Perhaps the most novel and important feature of this series of experiments is the evidence of the importance of the pharyngeal cavity for proteid digestion in this primitive form of vertebrate; and when we come to examine the histological structure of the alimentary tract we find that glandular secreting structures are more conspicuous in this part of the alimentary canal than in the intestine proper. In the pharynx the epithelium lining the body-wall and the adjacent branchial surfaces is undoubtedly different from the single layer of flattened epithelium cells which cover the lamellar folds of the branchiæ themselves. It is usually described as consisting of several layers; but very conspicuous amongst the small epithelium cells are numerous glandular looking cells, to which I have never found any reference in descriptions of this region. These cells are arranged in groups of five or six together, and correspond in height to the whole thickness of the epithelium; they are somewhat swollen in the middle and are covered superficially by the small epithelium cells with the exception of a small space above the centre of the group where their tips converge together at the surface. In some preparations a collection of deeply stained granules can be seen at the outer ends of these cells, and in others a stringy mucus-like secretion is issuing from them. It is striking how on the branchial folds of the anterior wall of the first gill-pouch the epithelium containing these cells predominates, only the most internal folds appearing to have retained their respiratory function. Clearly the abundance and the evident activity of these pharyngeal glands is a sufficient histological reason why the extract of the pharynx is so active in digestion. On the other hand the epithelium of the narrow anterior intestine consists of a single layer of tall columnar cells with a distinct cuticular border and uniformly ciliated; there is no histological evidence here of the presence of any secreting cells. The epithelium of the rest of the intestine is very uniform, and similar in character to that of the anterior intestine, though only the anterior dilated portion is ciliated; and the ciliation even here is not uniform, but occurs in patches. The whole structure of this region suggests an organ of absorption rather than of digestion.

The only glandular organ in connection with the intestine is the liver, with its large gall-bladder; in structure it is typically a tubular liver throughout, and there is no evidence of any differentiation of function in the cells.

Round the entrance of the bile-duct into the intestine are a few small glandular follicles which have received the name of pancreas, but in most specimens they are too few and too small for it to be possible that they could play any important part in digestion, at all events during the Ammocœtes-stage. After transformation the liver becomes completely separated from the alimentary canal, the duct becomes obliterated, and the liver itself undergoes fatty degeneration. At the same time the small glands in the wall of the intestine which have been called the pancreas increase in number and form quite a prominent ring. If these glands are pancreatic in function, it suggests itself that possibly the tryptic digestion may become more important than the peptic as the animal advances to the higher stage of evolution in the adult *Petromyzon*. I hope to be able to work out the digestive processes of the adult animal as soon as I can obtain material.

Finding the presence of a peptic ferment to be so general in all parts of the alimentary canal, the question arises if it is present also in other tissues of the animal. I have already mentioned that no sign of digestive activity can be obtained from extracts of the so-called thyroid, extracts of the central nervous system are also quite inactive, and extracts of muscle shew only the faintest signs of activity after many hours. Then I thought it might be of interest to test for peptic ferment in the skin as being an epithelial structure continuous with the lining of the pharynx and on account of its active secretion; and I found that a ·2 °/₀ HCl. extract of the skins of two Ammocœtes digested 1 c.c. fibrin in a little over 1 hr. The skin consists of several layers of cells, the most superficial being those which secrete. They are peculiar in having a very thick cuticular border, whose striated appearance is due to the presence of fine pores through which the secretion exudes. These cells are full of granules and when treated with methylene-blue in the living condition, as observed by Mr Hardy, the granules at the base of the cell stain blue, whilst those towards the surface appear rose-coloured in artificial light; this reaction he has every reason to believe indicates the presence of a zymogen.

As to the significance of this secretion, it does not seem probable that it could function in the digestion of food. I think it must have some important function, and considering the habits of the animal, which lies buried in the mud, Mr Hardy has suggested that this secretion of the skin may act as a protection against the attacks of bacteria and other organisms, which might otherwise be injurious.

(5) *On the Reaction of certain Cell-Granules with Methylene-Blue.* By W. B. Hardy, B.A., Gonville and Caius College.

[*Received November* 30, 1891.]

In 1878, Prof. Ehrlich pointed out the fact that the granule-containing cells of the body, whether found free in the body fluids or elsewhere, could be distinguished from one another by the character of the reaction of their granules with different aniline dyes*. He distinguished five classes of granules characterised by staining with acid, basic, or neutral dyes, or indifferently with acid or basic dyes (amphophil). The present communication deals with a further subdivision of the basophil granules into two groups, characterised by very distinct colour-reactions with the basic dye methylene-blue, the one class of granules staining a deep blue, the other a bright rose.

The distinction of tint depends, for some reason not at present at all obvious, on illumination with yellow artificial light. Under these circumstances the colour-contrast is one of extraordinary brilliancy. With daylight, or with gaslight after it has been filtered through neutral-tint glass, the rose colour either appears a blue like that shewn by the blue-staining granules, or is dulled to a blue-violet tint. The explanation of this change may be found in the fact that the yellow gaslight is relatively richer in red rays (or poorer in blue rays) than is daylight, or the phenomena may be of a more complex nature. Be this as it may, the abrupt transition from bright rose to bright blue produced by directing the mirror from the gas flame to the window is a striking feature of this colour-reaction.

The discrimination of basophil granules into rose-staining or blue-staining varieties may depend upon the dichroic nature of methylene-blue. Thin films produced by running a minute quantity of a strong solution under a coverslip appear rose with artificial light, while more dilute solutions appear blue.

Methylene-blue is a salt having the composition of a chloride, the base being a pigment of the aniline series, and it has already been noticed that alkalies produce a rose or reddish modification possibly by decomposing this salt and liberating the pigment-base. This suggests that the rose tint may not be solely a physical phenomenon but may depend upon a chemical action of the granule substance on the dye, or a chemical change produced by the osmosis of the pigment through the cell protoplasm to the granule.

That the reaction does not depend simply upon the thickness

* The various papers dealing with this subject have been republished by Dr Ehrlich in pamphlet form under the title "Farbenanalytische Untersuchungen zur Histologie und Klinik des Blutes." *Berlin*, 1891.

of the film of methylene-blue is shewn by the fact that the rose-tint may be developed by granules of sizes varying from mere points to spherules 2 to 4 μ in diameter. Nor does it depend upon the solidity or fluidity of the staining substance; for the contents of large vacuoles in the ectoderm cells of *Daphnia* frequently stain an intense rose, the rest of the cell appearing blue. Lastly the rose tint may be produced neither in the cell nor in the animal, but (in the case of *Daphnia*) by the action on the dye of a substance poured by the ectoderm cells into the surrounding water.

The hypothesis that the reaction is really of a chemical nature is favoured (1) by the general fact that the imbibition of dyes by the fresh unfixed cells is determined by the chemical nature of those dyes—whether the pigment be basic, or acid; and (2) by the peculiar method of imbibition of dyes by fresh cells. If still living cells, such as basophil blood corpuscles, are treated with either an organic fluid or normal salt solution, in which a small quantity of methylene-blue is dissolved it is noticed that imbibition of the dye is coincident with the onset of death. So long as the cell remains fully alive it resists infiltration by the pigment, and the granules remain uncoloured. This condition may, especially with eosinophil cells, last for hours. With the first onset of death the dye makes its way through the protoplasm and the granules become coloured. Later, when *rigor mortis* has become thoroughly established, the nucleus and cell body absorb the dye, and appear blue. In other words the first imbibition of the dye occurs at a period when the complex cell protoplasm is commencing to disintegrate, and when therefore profound chemical changes are taking place.

In order to determine whether granules are of the rose or blue staining varieties it is necessary to apply the stain in some relatively innocuous fluid to the living cells; and subsequent treatment with fixing reagents entirely obliterates the reaction. This is because all fixing agents with which I have experimented have some action on methylene-blue. Thus corrosive sublimate produces a rose-coloured modification, and converts blue staining into violet or rose. Ammonium picrate produces a violet tint except in the case of very intensely blue granules. Osmic acid converts the rose into a blue tint.

Rose-staining basophil granules have been found by me in free basophil cells of *Astacus*, and of Vertebrates, in the ectoderm of *Daphnia*, and of the Ammocœte larva of Lampreys, and in the alveolar cells of salivary glands.

The last two instances are of a specially suggestive nature, as affording instances of cells containing at the same time blue and rose-staining granules. In the cells lining the alveoli of the submaxillary gland of a rabbit I have seen, after treatment with dilute methylene-blue in normal salt solution, a zone of rose-coloured

granules surrounding the lumen and extending about half-way towards the basement membrane, while outside this there was a zone of blue-staining granules. This suggests that the rose-staining condition is a final stage in the elaboration of the constituents of the granules of these cells.

A still more instructive example is found in the ectoderm cells of the Ammocœte larva which I examined at the request of Dr Gaskell. These cells are each overlaid by a thick cuticle perforated by coarse canaliculi which lead from the cell protoplasm to the external surface of the animal. Miss Alcock has shewn that these cells, under appropriate stimuli, discharge on to the general surface a viscid substance which has the power of rapidly digesting fibrin in an acid medium. If these cells are treated with methylene-blue we find (1) that the extruded secretion gives the rose reaction, (2) that the pores in the cuticle may appear as rose-coloured rods, owing to their being filled with the secretion, and (3) that the cells themselves are occupied by rose-coloured granules which lie in the half of the cell next to the cuticular border, and by blue-coloured granules which occupy their deeper portions.

In the ectoderm of *Daphnia* rose-coloured *granules* are scanty, while, under certain circumstances to be detailed elsewhere, the cells may include a number of large vacuoles, the contents of which give a brilliant rose reaction. In connection with the presence of these vacuoles we find that *Daphnia* possesses the power of extruding on to its surface, through cuticular pores, a substance which swells up to form a jelly in water, and stains brilliant rose. This particular case will receive more detailed description on some future occasion. For the present I will only say that the secretion is used by the animal to prevent parasitic vegetable or animal growths obtaining a foothold on the shell.

I have never yet found a blood or lymph cell with both blue and rose-staining granules. It may be regarded as probable that blue-staining granules are absent from wandering cells. The cells with rose-staining granules have a remarkable distribution. In *Astacus*, as I have noted elsewhere*, they occur normally lodged in the spaces of a peculiar tissue which forms an adventitia to some of the arteries. They are only discharged into the blood as a result of special stimuli. In Vertebrates they occur to a marked extent in the peculiar adventitia of the blood-vessels of the spleen.

It is noticeable that I have so far failed to find rose-staining granules in endoderm cells, though I have examined the lining cells of the alimentary canal and of its glands in very diverse animal types.

The cells of the excretory organ (end-sac of *Daphnia*) contain granules which have a remarkable affinity for methylene-blue and stain a deep opaque blue.

* *Journal of Physiology*, 1892.

November 23, 1891.

PROFESSOR LEWIS IN THE CHAIR.

S. Skinner, M.A., Christ's College, was elected a Fellow of the Society.

The following Communications were made:

(1) *The Self-Induction of Two Parallel Conductors.* By H. M. MACDONALD, B.A., Clare College.

(*Abstract.*)

In Article 685, Maxwell's *Electricity and Magnetism*, Vol. II., the expression $\frac{1}{2}(\mu + \mu') + 2\mu_0 \log \frac{b^2}{aa'}$, is given for the self-induction per unit length of two parallel infinite cylindrical conductors, the radii of their sections being a, a', the distance between their axes b, μ, μ' their magnetic permeabilities, and μ_0 that of the surrounding medium. Lord Rayleigh remarked in the *Phil. Mag.* 1886, that this expression only holds when $\mu = \mu' = \mu_0$. To solve the general problem for steady currents it is necessary to know the conditions satisfied by the components of vector potential at the bounding surface of two media of different magnetic permeabilities; they are found to be

$$\frac{1}{\mu}\frac{\partial F}{\partial n} + \frac{1}{\mu'}\frac{\partial F}{\partial n'} = 0, \text{ etc.}$$

Transforming the equations by the relation

$$x + \iota y = c \tan \tfrac{1}{2}(\xi + \iota\eta),$$

taking $\eta = \alpha$, $\eta = -\beta$ as the bounding surfaces of the conductors, we have

$$\frac{\partial^2 H}{\partial \xi^2} + \frac{\partial^2 H}{\partial \eta^2} + \frac{4\pi\mu\omega}{(\cosh\eta + \cos\xi)^2} = 0, \quad \eta = \infty \text{ to } \eta = \alpha,$$

$$\frac{\partial^2 H_0}{\partial \xi^2} + \frac{\partial^2 H_0}{\partial \eta^2} = 0, \qquad \eta = \alpha \text{ to } \eta = -\beta,$$

$$\frac{\partial^2 H'}{\partial \xi^2} + \frac{\partial^2 H'}{\partial \eta^2} + \frac{4\pi\mu'\omega'}{(\cosh\eta + \cos\xi)^2} = 0, \quad \eta = -\beta \text{ to } \eta = -\infty,$$

$$H = H_0 \text{ and } \frac{1}{\mu}\frac{\partial H}{\partial \eta} = \frac{1}{\mu_0}\frac{\partial H_0}{\partial \eta} \text{ when } \eta = \alpha,$$

$$H_0 = H' \text{ and } \frac{1}{\mu_0}\frac{\partial H_0}{\partial \eta} = \frac{1}{\mu'}\frac{\partial H'}{\partial \eta} \text{ when } \eta = -\beta,$$

F and G being constant.

Solving these for H, H_0 and H' and determining L from them, we obtain

$$L = \tfrac{1}{2}(\mu + \mu') + 2\mu_0(\alpha + \beta)$$
$$+ \sum_1^\infty \frac{\left(1+\frac{\mu_0}{\mu}\right)e^{n(\alpha-\beta)} + \left(1+\frac{\mu_0}{\mu'}\right)e^{n(\beta-\alpha)} + \left(1-\frac{\mu_0}{\mu}\right)e^{-n(\alpha+3\beta)} + \left(1-\frac{\mu_0}{\mu'}\right)e^{-n(3\alpha+\beta)} - 4e^{-n(\alpha}}{n\left\{\left(1+\frac{\mu_0}{\mu}\right)\left(1+\frac{\mu_0}{\mu'}\right)e^{n(\alpha+\beta)} - \left(1-\frac{\mu_0}{\mu}\right)\left(1-\frac{\mu_0}{\mu'}\right)e^{-n(\alpha+\beta)}\right\}}$$

When $\mu' = \mu_0$ we have the case of the a conductor iron, the other being copper, and then

$$L = \tfrac{1}{2}(\mu + \mu_0) + 2\mu_0 \log \frac{b^2}{aa'} + 2\mu_0 \frac{\mu - \mu_0}{\mu + \mu_0} \log \frac{b^2}{b^2 - a^2},$$

the repulsive force between the conductors being

$$\frac{2\mu_0 I^2}{b}\left(1 - \frac{\mu - \mu_0}{\mu + \mu_0}\frac{a^2}{b^2 - a^2}\right).$$

Taking the case of conductors of equal section, the following table shews how the variable part of the coefficient of induction varies with their distance apart.

b	$\log \frac{b^2}{aa'}$	$\frac{\mu-\mu_0}{\mu+\mu_0} \log \frac{b^2}{b^2-a^2}$	Increase per cent.	$L - 50{\cdot}5$ Maxwell	$L - 50{\cdot}5$ from above
$2a$	1·38629	·282007	20·3	2·7725	3·3365
$3a$	2·19722	·11776	5·3	4·3944	4·6299
$4a$	2·77258	·06326	2·2	5·5451	5·6716
$5a$	3·21887	·039829	1·2	6·4377	6·5173
$6a$	3·58351	·027583	·7	7·1670	7·2221
$7a$	3·89164	·020211	·5	7·7832	7·8237
$8a$	4·15888	·015936	·3	8·3177	8·3496
$9a$	4·39425	·012131	·2	8·7885	8·8127
$10a$	4·60517	·009851	·2	9·2103	9·2300

The first column gives the distances between the axes of the conductors, the second the values of half the variable term in Maxwell's formula, the third half the term which has to be added to it, the fourth the increase per cent. of the variable part due to the term neglected by Maxwell, the fifth and sixth the values of the variable parts in both cases; μ_0 being taken unity and $\mu = 100$. The table shews that the term neglected is considerable when the conductors are close to one another, and decreases rapidly at first as b increases, afterwards slowly.

Again taking the conductors touching one another, the following table gives the maximum values of the correction as the radius of the iron conductor increases.

a	b	$\log \frac{b^2}{aa'}$	$\frac{\mu-\mu_0}{\mu+\mu_0} \log \frac{b^2}{b^2-a^2}$	Increase per cent.	$L-50{\cdot}5$ Maxwell	$L-50{\cdot}5$ from above formula
a'	$2a'$	1·38629	·282007	20·3	2·7725	3·3365
$2a'$	$3a'$	1·50407	·576147	38·3	3·0081	4·1604
$3a'$	$4a'$	1·67397	·810307	48·0	3·3479	4·9685
$4a'$	$5a'$	1·83257	1·001419	54·6	3·6651	5·6679
$5a'$	$6a'$	1·97407	1·162144	58·8	3·9481	6·2724
$6a'$	$7a'$	2·10005	1·300593	61·9	4·2001	6·8012
$7a'$	$8a'$	2·21297	1·422097	64·2	4·4259	7·2701
$8a'$	$9a'$	2·31447	1·530317	66·1	4·6289	7·6895
$9a'$	$10a'$	2·40794	1·627843	67·6	4·8159	8·0716
$10a'$	$11a'$	2·49320	1·716587	68·8	4·9864	8·4195

The first column expresses the radius of the iron conductor in terms of that of the other; the remaining columns are as in the first table.

The expression for the force can by suitably choosing a, a', b so that b is somewhat greater than $a\sqrt{2}$, be made to change sign, so that the force between the conductors would then be attractive.

In the case of two iron conductors

$$L = \mu + 2\mu_0 \left[\log \frac{b^2}{aa'} + \lambda \log \frac{b^4}{(b^2-a^2)(b^2-a'^2)} \right.$$

$$+ 2\lambda^2 \log \frac{b^2(b^2-a^2-a'^2)}{(b^2-a^2)(b^2-a'^2)}$$

$$\left. + \lambda^3 \log \frac{b^4(b^2-a^2-a'^2)^2}{(b^2-a^2)(b^2-a'^2)\{(b^2-a^2)^2-a'^2b^2\}\{(b^2-a'^2)^2-a^2b^2\}} + \text{etc.} \right],$$

where $$\lambda = \frac{\mu-\mu_0}{\mu+\mu_0}.$$

In this series the coefficients of the powers of λ rapidly diminish. It is perhaps worthy of remark that when μ is 100 or greater, the part depending on the size of the conductors and their distance apart is only slightly affected by the value of μ, as λ differs but little from unity.

(2) *The Effect of Flaws on the Strength of Materials.* By J. Larmor, M.A., St John's College.

The effect of an air-bubble of spherical or cylindrical form in increasing the strains in its neighbourhood was examined; and it was suggested that the results might be of practical service in drawing general conclusions as to the influence of local relaxations of stiffness of other kinds. In particular, it is shewn by the aid of the hydrodynamical form of St Venant's analysis, that a cavity of the form of a narrow circular cylinder, lying parallel to the axis of a shaft under torsion, will double the shear at a certain point of its circumference; and the effect of a spherical cavity will not usually be very different. For a cylindrical cavity of elliptic section, the shear may be increased in the ratio of the sum of its two axes to the smaller of them, this ratio becoming infinite in accordance with known theoretical principles for the case of a narrow slit. It is assumed in the analysis that the distance of the cavity from the surface of the shaft is considerable compared with its diameter, so that the influence of that boundary may be left out of account in an approximate solution[1].

The results will however also give the effect of a groove of semicircular or semi-elliptic section, running down the surface of the shaft, provided the curvature of the surface is small compared with the curvature of the groove.

(3) *The Contact Relations of certain Systems of Circles and Conics.* By W. McF. Orr, B.A., St John's College.

(*Abstract.*)

The following theorem is first established:—If four circles X, Y, P, Q, in a plane or on a sphere, are such that a circle can be drawn through one of each pair of intersections of X with P, X with Q, Y with P, and Y with Q respectively, (and therefore another circle through the remaining four intersections of the same circles,) the eight circles which touch X, Y and P, and the eight which touch X, Y and Q, can be arranged in sixteen groups of four circles, each group consisting of two touching X, Y and P, and two touching X, Y and Q, such that each group has two common tangential circles besides X and Y.

A similar result is of course true for groups of circles touching P, Q and X, and P, Q and Y respectively.

The above relation of condition is *triply* satisfied by the four circles that form any Hart-group of circles touching three others (restricting the title Hart-group to the eight groups that are analogous to the inscribed and escribed circles of a triangle).

[1] See *Phil. Mag.*, Jan. 1892.

Hence, taking as a particular case the inscribed and escribed circles of a plane or spherical triangle, the following result is obtained:—If we describe circles touching three by three the inscribed and escribed circles of a plane or spherical triangle, we obtain four sets of four circles, exclusive of the sides of the original triangle and its Hart-circle; each set forms a Hart-group and in addition we can obtain twenty-four groups of four circles by taking two out of any one set and two out of any other such that each group is touched by two circles besides the two circles they have been constructed to touch in common.

Any group whatever of four circles of the eight that touch any three given circles satisfy the above relation of condition, some singly, some doubly, and some triply; and by taking all such groups of four, and describing circles touching them three by three the following result is obtained:—Eight circles can be described to touch three given circles; these eight form fifty-six groups of three; to touch any three we can describe a set of four circles exclusive of the original three, and one which with them forms a group of four circles touching four others; and we can form a thousand and eight groups of four circles, two out of one set and two out of another, such that each group is touched by two common circles besides the two they have been constructed to touch in common.

These theorems are then extended to co-vertical cones which are either circular or have double contact with a given one, and by projection to conics having double contact with a given one.

In the last case one of the results obtained is:—If four straight lines X, Y, P, Q are such that through the intersections of X with P, X with Q, Y with P and Y with Q there can be described a conic having double contact with a given one ϕ, then the four conics touching X, Y and P and the four touching X, Y and Q, all having double contact with ϕ, can be arranged in eight groups each consisting of two touching X, Y and P, and two touching X, Y and Q, such that each group, besides touching X and Y, touch in common two tangent conics having double contact with ϕ; and similarly for conics touching P, Q and X, and P, Q and Y respectively.

The enunciation of the reciprocal theorem is obvious.

Another result is as follows:—Four conics can be described having double contact with a given one ϕ, and touching three given lines or passing through three given points; to touch any three of these four conics sixteen conics can be described having double contact with ϕ, exclusive of the original lines or points and four Hart-conics; there are thus four sets of sixteen conics. In addition to the groups of four touched by four conics having double contact with ϕ that can be formed by taking four out of the

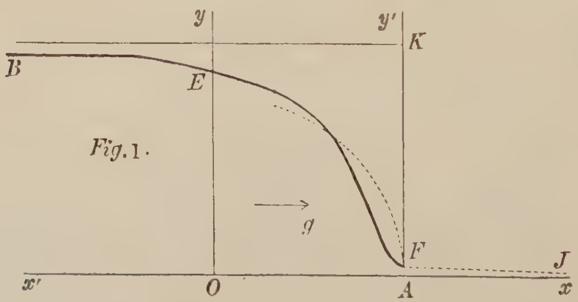
same set, there are thirty-two groups of four conics formed by taking two out of any one set and two out of any other, such that each group is touched by two conics having double contact with the given one ϕ, besides the two they have been constructed to touch in common, and as the four sets can be taken in pairs in six ways, there are thus one hundred and ninety-two such groups in all.

Some other theorems are obtained of a more complicated character.

The method of proof is purely geometrical and the first proposition, though not proved as shortly as it might have been, is made to depend mainly on a property of Bicircular Quartics given by Mr C. M. Jessop in the *Quarterly Journal of Mathematics*, Vol. XXIII., which is equally true for Sphero-Quartics, and which for the case of two circles may be enunciated a little differently as follows:—If X, Y are any two circles of the same family touching two given circles A and B, in a plane or on a sphere, and P, Q are any two circles of the other family touching A and B, then a circle can be drawn through one of each pair of the intersections of X with P, X with Q, Y with P, and Y with Q, and of course another circle through the remaining four intersections of the same pairs.

(4) *On Liquid Jets under Gravity.* By H. J. Sharpe, M.A., St John's College.

1. The motion (Fig. 1), which is in two dimensions, is supposed to be symmetrical with regard to $x'Ox$ which is the axis of the vessel and jet. BEF is the semi-outline of the vessel, FJ of the jet. AF is the semi-orifice, which is *small* compared with the dimensions of the vessel and the depth OA of the liquid. Gravity acts parallel to $x'Ox$. OE is the surface of the



liquid, which is maintained steady. AF is supposed to be so small that it may be considered either as the arc of a circle

with centre O in the surface of the liquid, or as a small straight line perpendicular to Ox. For simplicity we shall take OA the radius of the circle (or the depth of the liquid) as unity.

If g be the acceleration of gravity referred to this unit it will be convenient to put

$$a^2 \text{ for } 2g \text{ ....(1).}$$

We shall take O as the origin of Cartesian and Polar Coordinates x, y, r, θ and we shall put

$$x' \text{ for } (x-1) \text{ ....(2).}$$

Let χ be the stream function to the *right* of AF; u, v the velocities parallel to Ox, Oy.

Further let $$AF = \pi/p \text{..................................(3),}$$
where p is a *large* number.

On the *right* of AF we will take

$$\left.\begin{aligned} \frac{d\chi}{dy} &= u = ar^{\frac{1}{2}} \cos \tfrac{1}{2}\theta + \Sigma c_n' \epsilon^{-pnx'} \cos pny \\ -\frac{d\chi}{dx} &= v = -ar^{\frac{1}{2}} \sin \tfrac{1}{2}\theta + \Sigma c_n' \epsilon^{-pnx'} \sin pny \end{aligned}\right\} \text{......(4),}$$

where c_n' is an arbitrary constant. Σ indicates summation with regard to n for all integral values from 1 to ∞.

Therefore along AF, we have on the *right* of it at every point, *nearly*

$$\left.\begin{aligned} u &= a + \Sigma c_n' \cos pny \\ v &= -\tfrac{1}{2} ay + \Sigma c_n' \sin pny \end{aligned}\right\} \text{......................(5).}$$

[Of course AF is really half the small segment of a circle. The equations (4) and (5) are only approximate (the more so the larger p be taken) but it will be pointed out afterwards (Art. 5) how their exact forms can be found—forms which would be suitable for all values of p—but as these are rather complicated it is better to begin with the simplest case first.]

Let ψ be the stream function on the *left* of AF, and on the *left* of AF we will take

$$\left.\begin{aligned} \frac{d\psi}{dy} &= u = S\,(a_m \epsilon^{mx'} \cos my) + \Sigma c_n \epsilon^{pnx'} \cos pny + A \\ -\frac{d\psi}{dx} &= v = -S\,(a_m \epsilon^{mx'} \sin my) - \Sigma c_n \epsilon^{pnx'} \sin pny \end{aligned}\right\} \text{...(6),}$$

where a_m c_n and A are arbitrary constants, and S indicates summation with regard to m for a *finite* number of values of *m, the largest of which is supposed to be small compared with p.*

Therefore along AF we have on the *left* of it at every point

$$\left.\begin{aligned} u &= S(a_m \cos my) + \Sigma c_n \cos pny + A \\ v &= -S(a_m \sin my) - \Sigma c_n \sin pny \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(7).$$

But as the motion must be continuous through AF, the u's and v's in (5) and (7) must be the same. Therefore we get

$$\left.\begin{aligned} S(a_m \cos my) &= a - A + \Sigma(c_n' - c_n)\cos pny \\ -S(a_m \sin my) + \tfrac{1}{2}ay &= \Sigma(c_n' + c_n)\sin pny \end{aligned}\right\} \quad \ldots\ldots\ldots(8).$$

These must hold from $y = 0$ to $y = \pi/p$. But if we expand the left-hand sides of (8) by Fourier's Theorem, we can get c_n and c_n' as functions of n.

2. Doing so, we shall get

$$c_n' + c_n = 2S\left(a_m \sin\frac{m\pi}{p} \cdot \frac{n\pi\cos n\pi}{n^2\pi^2 - \dfrac{m^2\pi^2}{p^2}}\right) - \frac{a}{p}\,\frac{\cos n\pi}{n} \quad \ldots(9).$$

$$c_n' - c_n = -2S\left(a_m \cdot \frac{m\pi}{p}\sin\frac{m\pi}{p} \cdot \frac{\cos n\pi}{n^2\pi^2 - \dfrac{m^2\pi^2}{p^2}}\right) \quad \ldots\ldots\ldots(10).$$

Also $$A - a + S\left(a_m \times \frac{p}{m\pi}\sin\frac{m\pi}{p}\right) = 0 \quad \ldots\ldots\ldots(11).$$

Also in order that the second of the equations (8) may hold *at* the limit $y = \pi/p$, we must have

$$-S\left(a_m \sin\frac{m\pi}{p}\right) + \frac{a\pi}{2p} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(12).$$

We shall now prove an important property of c_n and c_n'.

As 1 is the least value of n, and as by hypothesis m/p is always a small fraction, we may always (if need be) safely expand the fractions in (9) and (10) in ascending powers of m^2/p^2n^2.

Doing so in (9), we shall get

$$c_n' + c_n = 2S\left\{a_m \sin\frac{m\pi}{p} \cdot \frac{\cos n\pi}{n\pi}\left(1 + \frac{m^2}{p^2n^2} + \&c.\right)\right\} - \frac{a}{p} \cdot \frac{\cos n\pi}{n} \quad \ldots\ldots\ldots(13).$$

But by (12) the first and last term here disappear, so that $(c_n' + c_n)$ is always a small quantity at most of the order $1/p^3$. Similarly from (10) we see that $(c_n' - c_n)$ and therefore c_n and c_n' are always small quantities at most of the order $1/p^2$. We say 'always'—even when $n = 1$ when they are largest.

3. From (6) the equation to BEF is

$$S\left(\frac{a_m}{m}\epsilon^{mx'}\sin my\right)+\Sigma\frac{c_n}{pn}\epsilon^{pnx'}\sin pny+Ay$$
$$=S\left(\frac{a_m}{m}\sin\frac{m\pi}{p}\right)+\frac{A\pi}{p}\dots\dots\dots\dots(14).$$

If $y=b$ is the equation to the asymptote to BEF

$$Ab=S\left(\frac{a_m}{m}\sin\frac{m\pi}{p}\right)+\frac{A\pi}{p}\dots\dots\dots\dots(15),$$

so that if b be finite, A is a small quantity of the order $1/p$.

Looking now at (6) we see that if OE is to be the surface of the liquid, u and v must, when $x'=-1$, be small quantities at most of the order $1/p$. A and the Σ term already satisfy that condition. In the S term m has several values, enough to satisfy conditions (11), (12) and (15). Suppose the particular m in (6) to be the *smallest* of these values, and suppose $m=\log p$, then when $x'=-1$, the S term also satisfies the surface condition, and the more accurately the larger p is, since $\log p/p$ diminishes, as p increases.

If FJ is to be a *jet* we must have, since AF is small, at every point of the jet, nearly

$$u^2+v^2=2gr.$$

But we see at once from (1) and (4) and Art. 2 that this condition is fulfilled, the error being of the order $1/p^2$.

4. To get some idea of the maximum value of this error, we see from (4), since at F we have nearly

$$u=a-c_1'$$

that $-2c_1'/a$ is a fair measure of this maximum.

From (10) and (13)

$$c_1'=\frac{1}{p^2}S(m^2a_m)\dots\dots\dots\dots(16).$$

From (11) and (12) we have nearly, since A is a small quantity,

$$-a+S(a_m)=0,\text{ and }\frac{a}{2}-S(ma_m)=0.$$

If for instance we take 8 and 9 for the two values of m, then p will be about 2981 and the maximum error about

$$+\cdot 0000143.$$

If we took a sufficient number of values of m to satisfy, in addition to previous conditions, the condition $S(m^2 a_m) = 0$, we see from (16) that the maximum error would be of the 3rd order of smallness, and so on for higher orders.

5. Suppose p instead of being large, were somewhat smaller, we should then proceed thus.

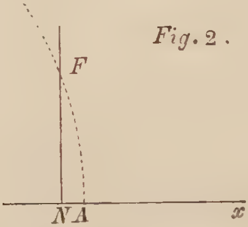


Fig. 2.

From F (fig. 2) draw FN perpendicular to Ax.

Let $NF = \pi/p'$ where

$$\frac{\pi}{p'} = \sin\frac{\pi}{p} \quad \text{.............................(17).}$$

In equations (4) to (8) &c. put p' for p, and consider (4) to apply to the *right*, and (6) to the *left* of NF. (8) also would have to hold from $y=0$ to $y=\pi/p'$. Of course from (17) p' could be expressed as accurately as desired in terms of p.

6. From (4) the equation to the outline FJ of the jet is

$$\frac{2a}{3} r^{\frac{3}{2}} \sin\frac{3}{2}\theta + \Sigma \frac{c_n'}{pn} \epsilon^{-pnx'} \sin pny = \frac{2a}{3} \sin\frac{3\pi}{2p} \quad \text{....(18).}$$

As the Σ term is of the order $1/p^3$, we see that in all solutions obtained by the present method the shape of the jet is nearly independent of the shape of the vessel, and is dependent only on the angle which the orifice subtends at O.

Upon this point light may be thrown by the following Article.

7. Since writing the preceding, I have examined somewhat carefully equation (14) which gives the outline of the vessel—in the case where m has the values 8 and 9. In this case a_m and a_n are determinate. I find that in (15) b is not perfectly arbitrary, but appears to have limits in order that the curve BEF may be continuous. I have tried to take it as large as possible. I have actually taken it $= 2\pi/9$ or about ·6981, but

whether this is the largest admissible value of b (for $m = 8$ and 9) I am not sure. The result is that I get a curve something of this shape (fig. 3) for BEF.

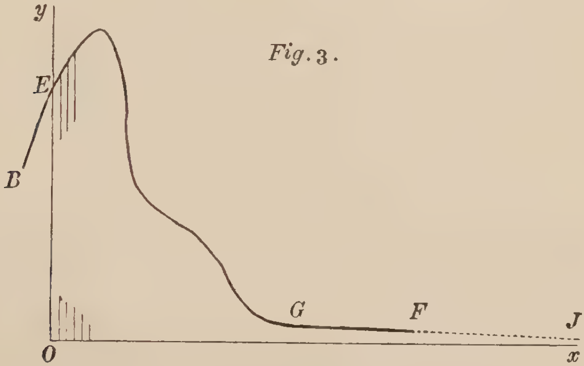


Fig. 3.

One noticeable feature is the existence of a long spout or pipe-like portion GF before we reach the orifice F, which may perhaps explain the result noticed in Art. 6.

I am afraid this spout exists in all solutions obtained by the present method. There is an interesting point about the curvature of the outer stream-line in the neighbourhood of F. It will be found from equations (14) and (18) that there is a sudden change of curvature at F—that the curvature on the right of F divided by curvature on left of F gives a small quantity of the order $1/p$, thus corroborating (as far as the approximate character of the present method will allow) a remark of Kirchhoff found at the end of Art. 96 of Lamb's *Motion of Fluids*.

(5) *Theory of Contact- and Thermo-Electricity.* By J. PARKER, M.A., St John's College.

In this paper, which treats of an electrified system of metals situated at rest in a vacuum in an unvarying state, we shall use the electromagnetic C.G.S. units, so that if Q, Q' be the charges of two small bodies at a distance of r centimetres, the electric repulsion between them is $\epsilon\dfrac{QQ'}{r^2}$ dynes, where ϵ is the constant 87×10^{19}.

We first require to know the energy U and entropy ϕ of our system. The system being necessarily so chosen that its total

charge is zero, let us suppose that by altering the relative positions of its parts, every one of its charges becomes zero, and denote the corresponding values of the energy and entropy by U_0 and ϕ_0. If from any cause, such as the parts of the system not being sufficiently numerous, this (or any other) operation cannot be directly performed, we may always make use of subsidiary bodies; and the final result being independent of the subsidiary bodies employed, we may argue as if they were entirely unnecessary.

For the energy U, we assume Helmholtz' expression

$$U = U_0 + \epsilon\Sigma \frac{QQ'}{r} + Q_A F_A + Q_B F_B + \ldots\ldots$$

$$= U_0 + \epsilon \{\tfrac{1}{2}Q_A V_A + \tfrac{1}{2}Q_B V_B + \ldots\ldots\} + Q_A F_A + Q_B F_B + \ldots\ldots (1),$$

where $A, B, C, \ldots$ are the different homogeneous bodies of uniform temperature which the system contains; Q_A, Q_B, Q_C, ... their charges; V_A, V_B, V_C, ... their potentials; and F_A, F_B, F_C, ... quantities which depend respectively on these bodies but not on their electric states. In what follows, the states of the different bodies will be completely defined by their temperatures, so that F will be a function of the temperature depending in form on the nature of the metal.

To obtain the value of ϕ, we make use of Joule's law that if a steady current I be flowing in a homogeneous body of uniform temperature and resistance R, the heat evolved is RI^2 ergs per second. From this we deduce that while a quantity Q is being transmitted, the heat evolved is RIQ; and therefore, since R is independent of I, as I diminishes, the heat evolved when a given finite quantity of electricity is transferred, diminishes in the same ratio as I. Now the process becomes more and more nearly reversible as I diminishes, but does not actually become reversible until I vanishes. Hence if our given system be made to undergo a reversible operation of any kind in which no part of it is compressed or distorted, and no charge made to pass from one body to a different body or to a body of the same kind but at a different temperature, there will be no thermal effect produced and consequently no change of entropy. So long therefore as the charges are not made to leave the bodies on which they were at first, the entropy of the system is unaltered by *any* change in the distribution of the charges or in the relative positions of the bodies. We may therefore put

$$\phi = \psi_A(Q_A) + \psi_B(Q_B) + \psi_C(Q_C) + \ldots\ldots,$$

where $\psi_A(Q_A)$ only depends on the body A and its charge, $\psi_B(Q_B)$ only on B and its charge, ...

Now take a second system identical with the first, forming with it a compound system whose entropy is 2ϕ. By a reversible process, such as we have already described, let a charge be made to pass from one metallic body A to the other body A, and suppose that no other charge passes from one body to another. Then, since by what precedes the entropy of the compound system is unchanged, it follows that, if q be the final charge of one body A and $2Q_A - q$ of the other, $\psi_A(q) + \psi_A(2Q_A - q)$ is independent of q, *whatever* q may be. We therefore infer that $\psi_A(Q_A) = \psi_A(0) + Q_A H_A$, where H_A is independent of Q_A.

Next, let us take a system formed of the original system ϕ and of a second system which also contains a metal body A at the same temperature as the body A in the first system but different in form and size and with any charge Q_A'. Let H_A' be the quantity corresponding to H_A. Then, by supposing any charge to pass from one metal A to the other metal A, without the passage of a charge between any other bodies, we find $H_A' = H_A$. Hence H_A is independent of the size and form of A as well as of its electric state.

Thus finally

$$\begin{aligned}\phi &= \psi_A(0) + \psi_B(0) + \ldots + Q_A H_A + Q_B H_B + \ldots \\ &= \phi_0 + Q_A H_A + Q_B H_B + \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).\end{aligned}$$

To complete the expressions for U and ϕ, we require an important identical relation which holds between F and H. Let the metal A and any other part which is at the same temperature θ_A be slowly heated to $\theta_A + d\theta_A$, and let the parts of the system be at the same time slowly moved about so that no charge passes from one body to another. Then we have

$$\left.\begin{aligned} dU &= dU_0 + \epsilon d\Sigma\frac{QQ'}{r} + Q_A dF_A + \ldots \\ d\phi &= d\phi_0 + Q_A\, dH_A + \ldots \end{aligned}\right\}.$$

If dW be the work done on the system, $dU - dW$ is the heat absorbed, and since the operation is reversible, we have

$$dU - dW = \theta_A d\phi,$$

or
$$dU_0 - \theta_A d\phi_0 - dW + \epsilon d\Sigma\frac{QQ'}{r} + Q_A dF_A + \ldots$$
$$- Q_A\theta_A dH_A - Q_B\theta_A dH_B - \ldots = 0.$$

Now take a second system identical with the first except that every charge is reversed in sign, and let it undergo a reversible

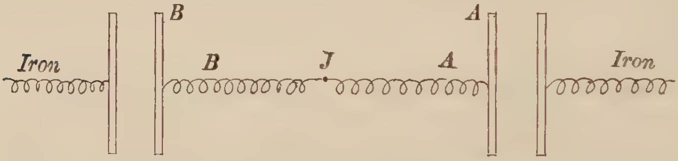
operation similar to that just described. Then since dU_0, $d\phi_0$, dW, and $\epsilon d\Sigma\frac{QQ'}{r}$ are the same as before, we obtain

$$dU_0 - \theta_A d\phi_0 - dW + \epsilon d\Sigma\frac{QQ'}{r} - Q_A dF_A - \dots$$
$$+ Q_A\theta_A dH_A + Q_B\theta_A dH_B + \dots = 0.$$

These two relations being true for all values of Q_A, Q_B, ..., it follows that

$$\frac{dF_A}{d\theta_A} = \theta_A\frac{dH_A}{d\theta_A}, \dots \qquad \text{(3)}.$$

To obtain the theory of the Peltier effect and the corresponding change of potential, let two long wires A, B, of different metals, be joined together at J, and also connected, as in the figure, with



two plates A, B, which are respectively of the same metals as the two wires. Parallel and opposite to these two plates place equal plates of any the same metal, as iron, and connect the iron plates by long iron wires with each other, or with a large distant mass of iron in the neutral state, so that the two iron plates are always at potential zero. Also, to make the calculations simple, let us suppose the air exhausted about the plates; but, to make the results general, the junction J must be surrounded by air, and to prevent the air coming near the plates, it must be enclosed in a bag and the junction J and the bag kept at a great distance from the plates. Then when the system is at a uniform temperature θ, if Q_B, Q_A be the charges of the plates; V_B, V_A the potentials; U and ϕ the energy and entropy of the system, we have

$$\left.\begin{aligned} U &= U_0 + \epsilon\{\tfrac{1}{2}Q_B V_B + \tfrac{1}{2}Q_A V_A\} + Q_B F_B(\theta) + Q_A F_A(\theta) \\ \phi &= \phi_0 + Q_B H_B(\theta) + Q_A H_A(\theta) \end{aligned}\right\} \qquad \text{(4)},$$

where U_0 and ϕ_0 will remain constant in what follows.

If now by slowly moving the plates B nearer together and slowly separating the plates A, any quantity of electricity q be made to pass slowly from A to B against the abrupt rise of potential $V_B - V_A$ at J, and the temperature of the system be kept constantly equal

to θ, there will be no thermal phenomenon in the system except at J, and the heat absorbed there will be θ times the increase of entropy, or $q\theta\,(H_B - H_A)$. This may be written qP_{BA}, and we see that $P_{BA} = -P_{AB}$, $P_{CA} = P_{CB} + P_{BA}$.

Again, the work done on the plates B is $-\frac{1}{2}q\epsilon V_B$, and the work done on the plates A, $\frac{1}{2}q\epsilon V_A$; so that the total work done on the system is $-\frac{1}{2}q\epsilon\,(V_B - V_A)$, or $-\frac{1}{2}qD_{BA}$, if D_{BA} stand for $\epsilon\,(V_B - V_A)$, or the electromotive force of contact. Hence, since the increase of energy is $\frac{1}{2}qD_{BA} + q\,(F_B - F_A)$, we have

$$\tfrac{1}{2}D_{BA} + F_B - F_A = -\tfrac{1}{2}D_{BA} + P_{BA},$$

or

$$D_{BA} + F_B - F_A = P_{BA} = \theta\,(H_B - H_A) \quad \text{.........(5).}$$

Combining equation (5) with the identity $\dfrac{dF}{d\theta} = \theta\dfrac{dH}{d\theta}$, we get

$$\left.\begin{aligned} &\frac{dD_{BA}}{d\theta} = H_B - H_A, \\ &\frac{d}{d\theta}\left(\frac{D_{BA}}{\theta}\right) = \frac{F_B - F_A}{\theta^2}, \\ &\frac{d}{d\theta}\left(\frac{P_{BA}}{\theta}\right) = \frac{dH_B}{d\theta} - \frac{dH_A}{d\theta} = \frac{1}{\theta}\frac{d}{d\theta}(F_B - F_A), \end{aligned}\right\} \quad \text{.......(6),}$$

and

$$P_{BA} = \theta\frac{dD_{BA}}{d\theta}.$$

The result $P = \theta\dfrac{dD}{d\theta}$ has been given on four independent occasions:—by Prof. J. J. Thomson in his Applications of Dynamics to Chemistry and Physics; by Maxwell in his small Treatise on Electricity, where he has abandoned the older assumption that $P = D$; by Duhem; and, lastly, by the present writer.

Next, let the plates A and B be of the same metal in the same molecular state but at slightly different temperatures θ, $\theta + d\theta$. Then if V and V' be potentials of A and B, Q and Q' their charges, we have

$$U = U_0 + \epsilon\,\{\tfrac{1}{2}Q'V' + \tfrac{1}{2}QV\} + Q'F(\theta + d\theta) + QF(\theta),$$

$$\phi = \phi_0 + Q'H\,(\theta + d\theta) + QH\,(\theta).$$

If therefore we suppose the change of temperature at J to be so gradual that the heat conducted across the junction while a small charge q passes slowly from one plate to the other, can be neglected, we easily find, if Σ be the 'specific heat of electricity,' that is, the coefficient of the Thomson effect, or $\Sigma d\theta$ the quantity

which corresponds to P_{BA} of equation (5),

$$\epsilon (V' - V) + \frac{dF}{d\theta} d\theta = \Sigma d\theta = \theta \frac{dH}{d\theta} d\theta.$$

$$\left.\begin{aligned} &\text{Hence} & \Sigma = \theta \frac{dH}{d\theta} - \frac{dF}{d\theta}, \\ &\text{and therefore} & V' - V = 0 \end{aligned}\right\} \quad \text{......................(7).}$$

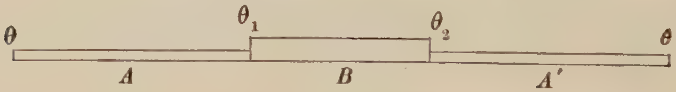
The result $V' - V = 0$, which asserts that there is *no* electromotive force of contact between two pieces of the same metal at different temperatures, is of the utmost importance. At first sight it may seem to be in contradiction to experiment; but on closer examination, as we shall shew later on, this is found not to be the case.

Equations (5), (6), and (7) contain the whole theory of the Peltier and Thomson effects. They enable us to discuss the properties of themoelectric circuits, and the results thus obtained include all those of Thomson and others which have been tested by experiment. After shewing this, we will point out the close analogy between our theory of thermoelectric circuits and Helmholtz' theory of the galvanic battery. Then we will consider some experiments bearing on our theory.

In the first place, we obtain, from equations (6) and (7), Thomson's result

$$\frac{d}{d\theta}\left(\frac{P_{BA}}{\theta}\right) = \frac{\Sigma_B - \Sigma_A}{\theta} \quad \text{........................(8).}$$

Next, let two pieces A, A', of the same kind of metal, be connected by a piece B of a different kind of metal, and let the free ends of A and A' be at the same temperature θ, while the junctions



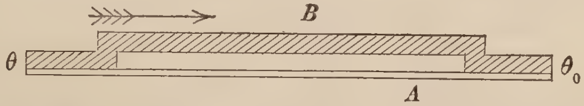
of B with A and A' are at the different temperatures θ_1, θ_2, respectively. Then since there is no electromotive force of contact between two pieces of the same metal at different temperatures, it follows that, in the state of equilibrium, the potential of the free end of A' will exceed that of the free end of A by

$$\frac{1}{\epsilon} \{D_{BA}(\theta_1) - D_{BA}(\theta_2)\}.$$

This will not generally be zero, and therefore, as experiment also shews, if a circuit be formed by joining the free ends of A and A', equilibrium will be impossible and there will be a current, called a thermoelectric current.

If A be a piece of metal in the same molecular state throughout whose temperature varies in any gradual way we please from end to end but has the same value at both ends, the ends will be at the same potential, and therefore if they be joined so as to form a circuit, there will be no current produced. This is the result obtained experimentally by Magnus, who found it impossible to obtain a current by unequal heating in a homogeneous circuit. If, however, we take a homogeneous circuit, and by filing make a junction of a very thick piece and a very thin piece, it was found by Maxwell that on applying a flame to this junction, a current is produced.

Now let a thermoelectric circuit be formed of two different metals A, B, as in the figure, and let the temperature of every



part of the circuit be kept constant; θ and θ_0 being the temperatures of the junctions. Then the 'electromotive force of the circuit' is defined to be ϵ times the sum of the abrupt rises of potential as we travel round the circuit in the direction of the current. Hence since the electromotive force of contact of two pieces of the same metal at different temperatures is zero, if the current be supposed to flow from A to B through the junction of temperature θ, the electromotive force E will be given by

$$E = D - D_0 \quad \text{.............................(9).}$$

For a circuit formed of several metals, we shall have

$$E = \Sigma D \quad \text{..................................(9)}'.$$

This result, which has not been tested directly by experiment, has been given by Duhem and assumed by Clausius.

When the current is steady, let I be its strength and R the resistance of the circuit. Then

$$E = RI,$$

since the sum of the abrupt rises of potential at the various junctions must be exactly balanced by the gradual fall of potential in the other parts of the circuit.

Confining ourselves, for the sake of simplicity, to the case of a circuit formed of only two metals, as in the preceding figure, the heat absorbed in a second at the junctions θ, θ_0, will be

$$\begin{aligned}(P - P_0)\, I &\equiv I\,[\{D + F_B(\theta) - F_A(\theta)\} - \{D_0 + F_B(\theta_0) - F_A(\theta_0)\}] \\ &\equiv I\,[E + \{F_B(\theta) - F_B(\theta_0)\} - \{F_A(\theta) - F_A(\theta_0)\}],\end{aligned}$$

which is exactly equal to the heat evolved in the rest of the circuit.

We can now give the principal results obtained by Sir W. Thomson, who avoids entirely the question of the electromotive forces of contact at the various junctions, either of two different metals, or of two portions of the same metal at different temperatures.

Combining equation (9) with the results

$$P = \theta \frac{dD}{d\theta} \text{ and } \frac{d}{d\theta}\left(\frac{P_{BA}}{\theta}\right) = \frac{\Sigma_B - \Sigma_A}{\theta},$$

we get Thomson's formulae

$$\left.\begin{aligned} E &= \int_{\theta_0}^{\theta} \frac{P}{\theta} d\theta = \int_{\theta_0}^{\theta} (H_B - H_A)\, d\theta, \\ P &= \theta \frac{dE}{d\theta}, \\ \frac{dE}{d\theta} &= \frac{dP_{BA}}{d\theta} - (\Sigma_B - \Sigma_A) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(10).$$

From the second of these equations, we see that if the current tends to cool any junction of two metals in passing through it, the electromotive force will be increased by raising the temperature of that junction. The electromotive force, as far as it depends on that junction, will be a maximum when P vanishes. As θ further increases, P will become negative and the electromotive force will diminish. The temperature T at which P vanishes is called the 'neutral point' of the two corresponding metals, or of the circuit when it is composed of these two metals only.

Again, when θ and θ_0 are so nearly equal that we may put $\theta_0 = \theta - \tau$, where τ is small, we have

$$D_0 = D - \tau \frac{dD}{d\theta} + \frac{\tau^2}{2} \frac{d^2 D}{d\theta^2} - \ldots,$$

and therefore for a circuit of two metals

$$E = \tau \frac{P}{\theta} - \frac{\tau^2}{2} \frac{d}{d\theta}\left(\frac{P}{\theta}\right) + \ldots.$$

When the θ junction is at the neutral point T, this gives Thomson's formula

$$E = -\frac{\tau^2}{2} \frac{1}{T} \frac{dP}{d\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

The current is then of the second order only and flows through the circuit in the same direction whether the other junction be hotter or colder than T.

When P does not vanish, we have another of Thomson's formulae

$$E = \tau \frac{P}{\theta} \quad\text{..........................(12).}$$

For simplicity taking τ to be positive, we see that E and P are of the same sign. The current therefore travels in such a direction as to absorb heat at the hotter junction of the two metals.

When the temperatures of the two junctions differ by a finite amount, it will be seen from (9) that the electromotive force is generally finite even when one junction is at the neutral point. If we suppose that the hot junction is at the neutral point, it is clear that both the Thomson effects cannot be absent, for then the heat that is developed in the homogeneous parts of the circuit would be all conveyed by the current, without assistance, from the coldest part of the circuit. It was the consideration of this case that led Sir W. Thomson to the discovery of the 'specific heat of electricity.'

If we had accepted the old assumptions that the thermal effects measure the electromotive forces of contact, or that $P = D$ and that the electromotive force of contact of two pieces of the same metal at slightly different temperatures is $\Sigma d\theta$, we should have had

$$E = P - P_0 - \int_{\theta_0}^{\theta} (\Sigma_B - \Sigma_A)\, d\theta \quad\text{...............(13).}$$

Now this is the very equation that is obtained by writing down the condition that the sum of the quantities of heat absorbed at the four junctions is equal to the heat evolved (according to Joule's law) in the rest of the circuit. Here then the two assumptions that $P = D$ and that the electromotive force of contact of two pieces of the same metal is $\Sigma d\theta$, exactly neutralize each other; but this fact, it is clear, does not prove that the assumptions are correct.

It may be noticed that the assumption $P = D$ cannot agree with our result $P = \theta \dfrac{dD}{d\theta}$ unless P is proportional to θ, which experiment shews is not usually the case. Again, if $\Sigma_B = \Sigma_A = 0$, equations (7) shew that F_B, H_B, F_A, H_A are constant, and therefore, by equation (6), $P = D = C\theta$, where C is independent of θ. But if $P = D = C\theta$, it does not follow conversely that F_B, H_B, F_A, H_A are constant, nor that $\Sigma_B = \Sigma_A = 0$.

Returning to our own theory, let us take as a first approximation

$$D = a + b\theta + c\theta^2,$$

where a, b, c are constants. Then

$$P = \theta(b + 2c\theta).$$

But P vanishes when $\theta = T$: hence

$$b + 2cT = 0.$$

Thus we have

$$\begin{aligned} E = D - D_0 &= b(\theta - \theta_0) + c(\theta^2 - \theta_0^2) \\ &= -2cT(\theta - \theta_0) + c(\theta^2 - \theta_0^2) \\ &= -2c(\theta - \theta_0)\left\{T - \frac{\theta + \theta_0}{2}\right\} \quad \ldots\ldots\ldots\ldots\ldots(14), \end{aligned}$$

the formula of Avenarius and Tait, which has been found to agree sufficiently well with experiment.

Again, since $$\frac{d}{d\theta}\left(\frac{D}{\theta}\right) = \frac{F_B - F_A}{\theta^2},$$

we find $$F_B - F_A = -a + c\theta^2,$$

which is satisfied by taking

$$F = k\theta^2 + k',$$

where k and k' are constants.

Hence $\dfrac{dF}{d\theta} = 2k\theta$, or the 'specific heat of electricity,' Σ, varies as the absolute temperature. Also $\dfrac{dH}{d\theta} = 2k$, or $H = 2k\theta + l$.

The 'thermoelectric diagram' of Prof. Tait is greatly simplified by our results. For if ω be the standard metal and M any other metal, the 'thermoelectric power' of M with reference to ω, or $\dfrac{dD_{M\omega}}{d\theta}$, is equal to $H_M - H_\omega$. The standard metal is taken to be lead, because the 'specific heat of electricity' of lead is zero, or H_ω constant. Hence in the diagram, the abscissa represents θ, and the ordinate, $H_M - H_\omega$, where H_ω is constant.

Lastly, let a galvanic battery have both poles of the same metal, and let every part of it be kept at the constant temperature θ. Let L be the heat evolved when we effect in any way at constant pressure the same chemical change as is produced by the passage through the battery of unit quantity of electricity in the direction in which the battery tends to give a current.

Also let us suppose that when the battery is in action, the chemical changes which take place are reversible, and the Peltier effects and electromotive force the same as when the current is infinitesimal. Then the heat absorbed at the junctions on the passage of unit quantity can be easily proved to be

$$\theta \frac{dE}{d\theta} \left(\text{just as } P = \theta \frac{dD}{d\theta}\right).$$

Hence since the heat evolved in the homogeneous parts in the same time is E, and the total heat evolved L, we have the result of Helmholtz and Gibbs,

$$L = E - \theta \frac{dE}{d\theta} \quad\text{..........................(15).}$$

It now only remains to look at the experimental evidence relating to the contact theory. In so doing we must remember that an electromotive force of contact is produced not only by the contact of two conductors, but by the contact of a conductor and a non-conductor, and also, but less easily, by the contact of two non-conductors. Again, for the sake of simplicity we shall often follow the custom of writing B/A for D_{BA}, and whenever it is necessary to indicate the temperature, we shall use suffixes: thus B/A_θ denotes that the two different bodies A, B, are at the same temperature θ; B_θ/B_{θ_0}, that two portions of the same substance are at the different temperatures θ, θ_0.

It has been pointed out by Maxwell that in experiments like those of Clifton and Pellat, in which two plates Z, C, of different metals, are employed in the open air, we really measure the sum $A/Z + Z/C + C/A$, or $D_{ZC} + A/Z - A/C$. In like manner, when we employ two plates of the same metal Z but at different temperatures θ, θ_0, we measure $Z_\theta/Z_{\theta_0} + A/Z_\theta - A/Z_{\theta_0}$. Now hitherto the terms $A/Z_\theta - A/Z_{\theta_0}$ have been always omitted, and the experiment has been supposed to prove that Z_θ/Z_{θ_0} is not zero. But clearly we cannot assume that

$$A/Z_\theta - A/Z_{\theta_0} = 0, \text{ or that } A/Z$$

is independent of the temperature; and therefore the experiment does not contradict our result that $Z_\theta/Z_{\theta_0} = 0$.

Let us assume that the thermal effects measure the electromotive forces of contact; and suppose that we repeat the experiments of Clifton or Pellat with two plates Z, C, of different metals, first at the temperature θ, and then at a slightly different temperature. From the results obtained, we find the value of

$$\frac{dP_{ZC}}{d\theta} + \frac{d}{d\theta}\{A/Z - A/C\},$$

which we will call $f_1(\theta)$. Next, if two plates of the same metal Z be employed but at slightly different temperatures, we obtain $\Sigma_z + \frac{d}{d\theta}(A/Z) =$ a known quantity $f_2(\theta)$. Similarly we may get

$$\Sigma_c + \frac{d}{d\theta}(A/C).$$

We then easily find

$$\frac{dP_{zc}}{d\theta} - (\Sigma_z - \Sigma_c) = f_1(\theta) - \{f_2(\theta) - f_3(\theta)\},$$

a result in which the influence of the air does not appear.

If we accept the thermo-dynamical theory here developed, these experiments give

$$\frac{dD_{zc}}{d\theta} + \frac{d}{d\theta}\{A/Z - A/C\} = f_1(\theta),$$

$$\frac{d}{d\theta}(A/Z) = f_2(\theta), \quad \frac{d}{d\theta}(A/C) = f_3(\theta),$$

and therefore

$$f_1(\theta) - \{f_2(\theta) - f_3(\theta)\} = \frac{dD_{zc}}{d\theta} = \frac{P_{zc}}{\theta}$$

$$= \frac{dP_{zc}}{d\theta} - (\Sigma_z - \Sigma_c),$$

which is the very result we obtained by assuming that the thermal effects measure the electromotive forces of contact. Hence experiments like those of Clifton and Pellat do not help us to decide which of the two theories is correct.

Again, on the *assumption* that $P = D$, it follows from the experiment which gives $D_{zc} + A/Z - A/C$, since P is very small, that the electromotive forces of contact between the plates and the air form great part of the phenomenon observed. It might be supposed that this conclusion could be tested directly by experiment by exhausting the air; but Pellat, by reducing the pressure to 1 cm. or 2 cms. of mercury, found little difference in the result.

If we take the thermo-dynamical theory, it will follow that in Pellat's experiment with two plates of the same metal at different temperatures, the *whole* of the phenomenon observed is due to the contact of the plates and air. In the case of two plates of different metals at the same temperature, a large part of the phenomenon observed will also probably be due to the contact of the metals with the air. Taking into account the

experiments of Brown and Pellat on the effects of reducing the pressure of the air, we conclude that within attainable limits of pressure, A/Z, the electromotive force of contact of the metal and air is nearly independent of the pressure of the air. It will now be shewn directly from theory not to be unreasonable to suppose that this may be the case.

If, for shortness, we take A/Z to be a volt, or 10^8 times the electromagnetic C. G. S. unit, and suppose the distance, d, of the two electric layers to be $\frac{1}{3 \times 10^8}$ cm., the attraction between the metal and the air per square centimetre will be

$$\frac{1}{8\pi\epsilon^2}\left(\frac{A/Z}{d}\right)^2 \text{ dynes,}$$

or about $40{,}000 \times 10^6$ dynes. Hence since the pressure of one atmo is only about 10^6 dynes per square centimetre, we can partly understand why the reduction of the pressure to 1 cm. or 2 cms. of mercury produces so little effect.

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PROCEEDINGS

OF THE

Cambridge Philosophical Society.

Monday, Feb. 8, 1892.

Professor Darwin, President, in the Chair.

Mr W. Heape, M.A., Trinity College, was elected a Fellow of the Society.

The following communication was made to the Society:

Long Rotating Circular Cylinders. By C. Chree, M.A., Fellow of King's College.

§ 1. In Vol. VII., Pt IV., pp. 201—215 of the *Proceedings* I found a solution for a thin elastic solid disk of isotropic material rotating with uniform angular velocity about a perpendicular to its plane through its centre. In the present paper the same method is applied to a long right circular cylinder of isotropic material rotating about its axis. The cross section of the cylinder when solid is supposed of radius a, when hollow its outer and inner boundaries are of radii a and a' respectively. The axis of the cylinder is taken for axis of z, the origin being at the middle point, and the notation for the displacements, strains, stresses, etc. is the same as in my previous paper, except that the dilatation is denoted by Δ.

If Poisson's ratio, η, be zero the solution obtained here satisfies all the internal and all the surface equations, whatever be the ratio of the length, $2l$, of the cylinder to its diameter. But for other values of η it is only true to the same degree of approximation as Saint-Venant's solution for beams, and like that solution can legitimately be applied only when l/a is large. This restriction of Saint-Venant's solution, whether for torsion or flexure, is not perhaps in general sufficiently recognised, but the best authorities I believe regard it as sufficiently exact only when the

length of the beam attains to something like ten times its greatest diameter. Unless $\eta = 0$ the present solution ought to be similarly restricted, and it should not be applied to the portions of the cylinder immediately adjacent to its ends. The solution considers solely the action of the "centrifugal force", taking no account of gravity or of the action of any forces applied at the ends by the bearings.

§ 2. The solution satisfies exactly the internal equations—viz. (2) and (3) *l.c.* p. 203—, and also the conditions at the curved surface or surfaces. The only condition it fails to satisfy exactly is that $\widehat{zz}$ vanish at every point of the flat ends. Instead of this we have to be content with satisfying

$$\int_{a'}^{a} 2\pi r\, \widehat{zz}\, dr = 0,$$

i.e. instead of making the normal stress over every element zero we make the resultant normal stress zero. The solution is thus based on the principle of *statically equivalent load systems*—referred to in my previous paper, pp. 206–7—and so can be regarded as satisfactory only when the dimension $2a$ is small compared to the dimension $2l$.

The solution is found by starting with the expressions (16), p. 205, for the displacements, which satisfy the internal equations. We have then to determine the arbitrary constants by means of the surface conditions, viz.,

$$\widehat{rz} = 0 = \widehat{rr}, \text{ when } r = a \text{ and when } r = a',$$

$$\widehat{rz} = 0 = \int_{a'}^{a} 2\pi r\, \widehat{zz}\, dr \text{ when } z = \pm l.$$

§ 3. It is unnecessary to reproduce the algebraical work, as the solution may easily be verified. In terms of Young's modulus E and Poisson's ratio η it is as follows:

$$u = \omega^2\rho\,(a^2 + a'^2)\, r\, \frac{3 - 5\eta}{8E(1-\eta)} - \omega^2\rho r^3 \frac{(1-2\eta)(1+\eta)}{8E(1-\eta)} + \omega^2\rho\, \frac{a^2a'^2}{r}\, \frac{(1+\eta)(3-2\eta)}{8E(1-\eta)} \quad \text{.........(1)},$$

$$w = -\,\omega^2\rho\,(a^2 + a'^2)\, z\, \frac{\eta}{2E} \quad \text{.........(2)},$$

$$\Delta = \omega^2\rho\,(a^2 + a'^2)\, \frac{(1-2\eta)(3-\eta)}{4E(1-\eta)} - \omega^2\rho r^2 \frac{(1-2\eta)(1+\eta)}{2E(1-\eta)} \quad \text{....(3)},$$

$$\widehat{rr} = \omega^2\rho\,(a^2 - r^2)(1 - a'^2/r^2)\, \frac{3 - 2\eta}{8(1-\eta)} \quad \text{.........(4)},$$

$$\widehat{\phi\phi} = \omega^2\rho\,(a^2 + a'^2 + a^2a'^2/r^2)\,\frac{3-2\eta}{8(1-\eta)} - \omega^2\rho r^2\,\frac{1+2\eta}{8(1-\eta)} \quad \text{..........(5)},$$

$$\widehat{zz} = \omega^2\rho\,(a^2 + a'^2 - 2r^2)\,\frac{\eta}{4(1-\eta)} \quad \text{..................................(6)},$$

$$\widehat{rz} = 0 \quad \text{..(7)}.$$

For a solid cylinder the displacements and stresses may be correctly deduced from the above by omitting all the terms containing a'^2. This solution for a solid cylinder is identical with one deduced from the case of a rotating spheroid* by supposing the ratio of the axis of figure to the perpendicular axis to increase indefinitely.

§ 4. In what follows I shall assume 0 and ·5 as limiting values of η†. As regards the latter limit there is a difficulty that will be best understood by reference to the relations

$$n/m = 1 - 2\eta,\ 3n\,(1 - n/3m) = E$$

between E, η and Thomson and Tait's elastic constants.

If we regard E as finite, we must when $\eta = \frac{1}{2}$ have m infinite. Thus those terms in the expressions for the stresses in terms of the strains which contain m as a factor may remain finite, though the corresponding strains vanish in the limit when $\eta = \frac{1}{2}$. This explains the apparent inconsistency in the expressions supplied by our solution for this value of η. The strains in this case in the solid cylinder take the remarkably simple forms

$$u = \omega^2\rho a^2 r/8E,\ w = -\,\omega^2\rho a^2 z/4E,\ \Delta = 0 \quad \text{.........(8)};$$

so that the three principal strains, viz. $\frac{du}{dr}$, u/r and $\frac{dw}{dz}$, are everywhere constant. The vanishing of Δ is a necessary consequence of m being infinite, for this implies that the material is incompressible.

§ 5. The expressions for the strains and stresses in the axis of a solid cylinder and at the inner surface of a hollow cylinder in which a'/a is infinitely small are, it will be noticed, totally different; for in the former case terms in a'^2/r^2 simply do not exist, whereas in the latter case $a'^2/r^2 = 1$. There is thus, as in the thin disk, a discontinuity in passing from a solid to a hollow cylinder however small a'/a may be. At first sight this appears absurd, for it may be argued that if matter has a molecular structure, as is generally supposed, then cavities exist everywhere between the molecules, and there is no reason why a cylinder

* *Quarterly Journal...*, vol. XXIII., 1889, p. 23, Equation (103).

† *Phil. Mag.* September 1891, pp. 235—6.

apparently solid should not have a cavity or cavities devoid of molecules occupying the whole or a great portion of its axial length. Would not then, it might be urged, such a solid cylinder act according to our solution quite differently from another of the same material in whose axial line there happened to be numerous molecules? The following seems a satisfactory explanation of this difficulty:

A *hollow* cylinder in the mathematical sense is one in which $\widehat{rz}$ and $\widehat{rr}$ vanish over $r = a'$; but this implies that a', even when infinitely small compared to a, is still so great compared to molecular distances, that the action of molecules separated by a distance of order a' is inappreciable. There is thus no sudden discontinuity, as our solution seems to imply, but a gradual transition as a' increases from being a molecular distance to being a distance so great that the mathematical conditions for a free surface are satisfied. This transition stage is not within the compass of the present mathematical theory; but this is hardly a matter of practical importance, for the mathematical conditions are probably satisfied in any existing hollow cylinder as exactly over its inner as over its outer surface.

For brevity, $a'/a = 0$ will be employed to denote the cylinder that is hollow in the mathematical sense, but in which a'/a is extremely small. In the same way $a'/a = 1$ will be employed to denote a cylinder whose wall thickness $a - a'$ is extremely small compared to a, though great compared to molecular distances.

§ 6. Returning to our solution we see that $\widehat{zz}$ vanishes when $\eta = 0$, so that all the surface conditions are then exactly satisfied. The solution in this case is thus complete and applies to circular cylinders of all shapes, to thin disks as well as to very long cylinders. When η is small $\widehat{zz}$ is very small compared to the greatest stress $\widehat{\phi\phi}$, and even when η is $\frac{1}{4}$ the greatest value of $\widehat{zz}$ bears to the greatest value of $\widehat{\phi\phi}$ a ratio which is $\frac{1}{5}$ in a solid cylinder and not more than $\frac{1}{10}$ in a hollow cylinder. When, however, η is $\frac{1}{2}$ the greatest value of $\widehat{zz}$ in a solid cylinder is one half the greatest value of $\widehat{\phi\phi}$, and so may be by no means a small stress. Now in the case of the thin disk the stresses which failed to vanish over the edges were always very small compared to the largest stresses. Thus, to all appearance, our solution for long cylinders is not quite so satisfactory as that for thin disks unless Poisson's ratio be small.

§ 7. A striking difference between the effects of rotation on thin disks and on long cylinders in which η is not zero, is that whereas in the former case originally plane sections perpendicular to the axis of rotation become paraboloidal, in the latter case

such sections remain plane. Unfortunately this absence of curvature could hardly be observed except at the ends, where our solution cannot strictly be applied.

The shortening of the cylinder per unit of length is given by

$$(-\delta l/l) = (-w/z) = \omega^2\rho\,(a^2 + a'^2)\,\eta/2E \quad \ldots\ldots\ldots(9),$$

with $a'^2 = 0$ for the solid cylinder. Though it vanishes with η, this is in ordinary materials an important alteration. Its magnitude in terms of $\omega^2\rho a^2/E$—a quantity depending on the density, Young's modulus and the velocity at the outer surface—is recorded in the following table for the value ·25 of η, for various values of a'/a:—

TABLE I.

Shortening of cylinder per unit of length, $\eta = {}$·25.

$a'/a =$	0	·2	·4	·6	·8	1
$(-\delta l/l) \div (\omega^2\rho a^2/E) =$	·125	·13	·145	·17	·205	·25

The entry under $a'/a = 0$ applies also to the solid cylinder. Since the shortening varies directly as η, its amount in terms of $\omega^2\rho a^2/E$ may be at once written down for any other value of η. Numerical measures of $(-\delta l/l)$ in two typical cases will be found in Tables IX. and XI.

§ 8. The other displacements of most interest are the alterations δa and $\delta a'$ in the radii of the two cylindrical surfaces. The ratios of these alterations to the original lengths are given in terms of $\omega^2\rho a^2/E$ by the formulae

$$(\delta a/a) \div (\omega^2\rho a^2/E) = \tfrac{1}{4}\{1 - \eta + (3 + \eta)\,a'^2/a^2\} \ldots\ldots(10),$$

$$(\delta a'/a') \div (\omega^2\rho a^2/E) = \tfrac{1}{4}\{3 + \eta + (1 - \eta)\,a'^2/a^2\} \ldots\ldots(11).$$

Thus the radii of both surfaces are always increased. Taking $\omega^2\rho a^2/E$ as constant, the following table shews how these alterations of the radii vary with η and with a'/a:—

TABLE II.

Value of $(\delta a/a) \div (\omega^2\rho a^2/E)$.

η $a'/a =$	0	·2	·4	·6	·8	1·0
0	·25	·28	·37	·52	·73	1·0
·25	·1875	·22	·3175	·48	·7075	1·0
·5	·125	·16	·265	·44	·685	1·0

TABLE III.

Value of $(\delta a'/a') \div (\omega^2 \rho a^2/E)$.

η $a'/a=$	0	·2	·4	·6	·8	1·0
0	·75	·76	·79	·84	·91	1·0
·25	·8125	·82	·8425	·88	·9325	1·0
·5	·875	·88	·895	·92	·955	1·0

The numerical results in these two tables are *exact**. Both $\delta a/a$ and $\delta a'/a'$ are linear in η; thus results for other values of η are easily and accurately supplied by interpolation. The results under $a'/a=0$ apply also to the solid cylinder in Table II., but not of course in Table III. The formulae (10) and (11) are identical with (45) and (46), *Proceedings*, *l.c.* p. 213, which give δa and $\delta a'$ for a thin disk. Thus Tables II. and III. apply also to thin disks.

The large and steady increase in the value of $\delta a/a$ as a'/a increases is very conspicuous. It is also noteworthy that when a'/a has a given value, $\delta a'/a'$ increases but $\delta a/a$ diminishes as η increases. In fact by (10) and (11)

$$\delta a/a + \delta a'/a' = (\omega^2 \rho a^2/E)(1 + a'^2/a^2),$$

so that $\delta a/a + \delta a'/a'$ is independent of η. When a'/a approaches 1 the influence of η on the magnitude of the alterations in the radii tends to disappear. An idea of the numerical magnitude of $\delta a/a$ and $\delta a'/a'$ will be most easily derived from the special cases treated in Tables IX. and XI.

For the alteration $\delta a - \delta a'$ in the wall thickness we have the same formula as for a thin disk, viz. (47), p. 213, and this thickness is increased or diminished by rotation according as

$$a'/a < \text{ or } > (1 - \sqrt{\eta}) \div (1 + \sqrt{\eta});$$

see *Proceedings, l.c.* Equation (48), p. 213, and subsequent remarks.

Comparing Tables I., II. and III. it will be seen that $(-\delta l/l)$ is by no means negligible compared to $\delta a/a$ and $\delta a'/a'$ unless η be small. Thus as a/l is small, the shortening of the cylinder should in general be more easily detected than the alterations in its radii.

§ 9. Since $\widehat{rz}$ is zero the principal strains are everywhere

$$\frac{dw}{dz}, \ \frac{du}{dr} \text{ and } \frac{u}{r}.$$

* This term as applied to this and following tables means that the numerical results are as exact as the formulae, and not merely the first figures of a decimal.

The *longitudinal strain*, $\frac{dw}{dz}$, is everywhere negative, *i.e.* a compression. It has the same constant value as w/z, and so is given in Table I. The *transverse strain*, u/r, is everywhere positive, *i.e.* an extension, and is never algebraically less than the *radial strain* $\frac{du}{dr}$. Its greatest value, $\bar{s}$, which is found at the axis of a solid cylinder or the inner surface of a hollow cylinder, is thus the *greatest strain*. For a hollow cylinder it is the quantity $\delta a'/a'$ of equation (11) and Table III. In a solid cylinder it is given by

$$\bar{s} = (u/r)_{r=0} = \frac{\omega^2 \rho a^2}{E} \frac{3-5\eta}{8(1-\eta)} \qquad \text{..............(12)};$$

whence answering to $\eta = 0$, $\eta = \cdot 25$ and $\eta = \cdot 5$ we obtain

$$\bar{s} \div (\omega^2 \rho a^2/E) = \cdot 375, \quad \cdot 291\dot{6} \text{ and } \cdot 125 \text{ respectively.}$$

The radial strain is most conveniently dealt with by means of the formula

$$\frac{8E(1-\eta)}{\omega^2 \rho} r^2 \frac{du}{dr} = f(r) \qquad \text{..................(13)};$$

where for a solid cylinder

$$r^{-2} f(r) = a^2 (3-5\eta) - 3r^2 (1-2\eta)(1+\eta) \qquad \text{......(14}a\text{)},$$

and for a hollow cylinder

$$f(r) = -a^2 a'^2 (1+\eta)(3-2\eta) + (a^2 + a'^2) r^2 (3-5\eta) \\ - 3r^4 (1-2\eta)(1+\eta) \qquad \text{......(14}b\text{)}.$$

For the sign of $\frac{du}{dr}$ we need only consider that of $f(r)$.

§ 10. In a solid cylinder $f(r)$ is positive inside and negative outside the surface

$$r^2 = a^2 (3-5\eta) \div \{3(1-2\eta)(1+\eta)\} \qquad \text{............(15)};$$

but the radius of this surface exceeds a when $\eta > \dot{3}$.

Thus in a solid cylinder when $\eta > \cdot\dot{3}$ the radial strain is everywhere an extension; when, however, $\eta < \cdot\dot{3}$ there is a cylindrical surface, viz. (15), outside of which it is a compression. When $\eta = 0$ or $\cdot\dot{3}$ the radial strain vanishes over the surface of the cylinder, and elsewhere is an extension; but for intermediate values of η the region wherein this strain is a compression has a small but finite thickness. For a given value of a this thickness has a maximum value of $\cdot 03775a$ approximately when $\eta = \cdot 2$.

§ 11. The variations in the sign of $\frac{du}{dr}$ in a hollow cylinder may be most easily investigated by means of the equation

$$f(r) = 0 \qquad\qquad (16),$$

where $f(r)$ is given by (14*b*), regard being had to the sign of the surface values of $f(r)$, viz.:

$$f(a') = -2\eta a'^2 \{(3-\eta)a^2 + (1-3\eta)a'^2\} \qquad\qquad (17),$$

$$f(a) = -2\eta a^2 \{(1-3\eta)a^2 + (3-\eta)a'^2\} \qquad\qquad (18).$$

Let us denote by a_2'/a the least positive root, when real, of

$$(x^2+1)^2(3-5\eta)^2 - 12x^2(1+\eta)^2(1-2\eta)(3-2\eta) = 0 \ldots (19),$$

and by a_1'/a the positive root of

$$x^2 = (3\eta-1)/(3-\eta) \qquad\qquad (20).$$

Then a_2'/a is that value of a'/a for which $f(r)$ has equal roots, and a_1'/a is that value for which $f(a)=0$. When $a'/a < a_1'/a$ then $f(a)$—and so $\frac{du}{dr}$ at the outer surface—is positive. It is obvious that $(a')^{-2}f(a')$—and so $\frac{du}{dr}$ at the inner surface—is negative for all permissible values of η greater than 0. For the limiting case a'/a infinitely small, one root of (16) is of order a'. This root is given, neglecting higher powers of a'/a, by

$$r_1^2/a^2 = (a'^2/a^2)(1+\eta)(3-2\eta)/(3-5\eta) \ldots\ldots (21).$$

Though for shortness we refer to this case as that where $a'/a = 0$, it is most convenient not to neglect the vanishingly small thickness $r_1 - a'$ within which $\frac{du}{dr}$ is negative, as we are thus enabled to include this case under the general classification. The phenomena may then be grouped under four classes, according to the value of η, with transition cases.

Class I., $\eta = 0$.

Here $(a')^{-2}f(a') = 0 = f(a)$, and $f(r)$ is positive for all intermediate values of r. Thus for all values of a'/a, the radial strain vanishes over both surfaces of the cylinder, and elsewhere is an extension.

Class II., $0 < \eta < \cdot\dot{3}$.

Here a_1'/a is imaginary.

Sub-class (i), $a'/a < a_2'/a$:

$f(r)$ vanishes over two distinct surfaces within the cylinder. Between these the radial strain is an extension, elsewhere it is a compression.

Transition case, $a'/a = a_2'/a$:

The two surfaces over which $f(r)$ vanishes coincide. At this surface the radial strain vanishes, elsewhere it is a compression.

Sub-class (ii), $a'/a > a_2'/a$:

The radial strain is everywhere a compression.

Transition case to Class III., $\eta = \cdot\dot{3}$:

This follows the same laws as Class II., except that for $a'/a = 0$ one of the two surfaces over which $f(r)$ vanishes coincides with the outer surface of the cylinder, so that there is not, as in sub-class (i) above, a volume of finite thickness at the outer surface wherein the radial strain is a compression.

CLASS III., $\cdot\dot{3} < \eta < (4 - \sqrt{7})/3$, *i.e.* $\cdot 4514$ approximately.

Here a_1'/a is real.

Sub-class (i), $a'/a < a_1'/a$:

$f(r)$ vanishes over one surface within the cylinder, and the radial strain is a compression within, an extension outside of this surface.

Transition case, $a'/a = a_1'/a$:

$f(r)$ vanishes over a second surface, but this coincides with the outer surface of the cylinder.

Sub-class (ii), $a_1'/a < a'/a < a_2'/a$:

$f(r)$ vanishes over two distinct surfaces within the cylinder. Between these the radial strain is an extension, elsewhere it is a compression.

Transition case, $a'/a = a_2'/a$:

The two surfaces over which $f(r)$ vanishes coincide within the cylinder. At this surface the radial strain vanishes, elsewhere it is a compression.

Sub-class (iii), $a'/a > a_2'/a$:

The radial strain is everywhere a compression.

Transition case to Class IV.

$$\eta = (4 - \sqrt{7})/3\,; \quad a_1'/a = a_2'/a = \cdot 3728 \text{ approximately.}$$

This follows the same laws as Class III., except that Sub-class (ii) is not represented, and for $a'/a = \cdot 3728$ the radial strain vanishes over two surfaces both coinciding with the outer surface of the cylinder.

Class IV., $(4-\sqrt{7})/3 < \eta \not> 5$.

Sub-class (i), $a'/a < a_1'/a$:

$f(r)$ vanishes over one surface within the cylinder, and the radial strain is a compression within, an extension outside of this surface.

Transition case, $a'/a = a_1'/a$:

The radial strain vanishes over the outer surface of the cylinder and elsewhere is a compression.

Sub-class (ii), $a'/a > a_1'/a$:

The radial strain is everywhere a compression.

§ 12. As illustrating Class II. and the transition to Class III. we shall consider the values ·25 and $\cdot\dot{3}$ of η. The approximate values of a_2'/a in these two cases are respectively ·4276 and ·3728*; *i.e.* the radial strain is everywhere a compression when, η being ·25, a'/a exceeds ·4276, and when, η being $\cdot\dot{3}$, a'/a exceeds ·3728. When $a'/a < a_2'/a$ the radial strain vanishes within the cylinder over two surfaces whose radii r_1 and r_2 are the positive roots of (16). The values of r_1/a and r_2/a for a series of values of a'/a are given in the following table to three places of decimals:—

Table IV.

Radii of surfaces over which $\dfrac{du}{dr} = 0$; $\eta = \cdot 25$ *and* $\cdot\dot{3}$.

	$a'/a =$	0	·1	·2	·3	·4
$\eta = \cdot 25$	$r_1/a =$	$1\cdot 336a'/a$	·134	·273	·423	·616
	$r_2/a =$	·966	·962	·947	·916	·839
$\eta = \cdot\dot{3}$	$r_1/a =$	$1\cdot 528a'/a$	·154	·315	·5	
	$r_2/a =$	1·0	·993	·970	·9165	

The radial strain is a compression at points whose axial distance lies between a' and r_1 or between r_2 and a, an extension where the axial distance lies between r_1 and r_2.

§ 13. As illustrating Class III. we shall consider the value ·4 of η. For this we find approximately

$$a_1'/a = \cdot 27735,\quad a_2'/a = \cdot 3486.$$

Thus when a'/a is less than ·27735 the radial strain vanishes over only one surface within the cylinder, being a compression within, an extension outside of this surface. When $a'/a = \cdot 27735$ the radial strain vanishes over two surfaces, but one of these is the outer surface of the cylinder. When a'/a lies between ·27735

* Equation (19) has the same roots when $\eta = \cdot\dot{3}$ as when $\eta = (4-\sqrt{7})/3$.

and ·3486 the radial strain vanishes over two distinct surfaces, of radii r_1 and r_2, within the cylinder, being an extension between these surfaces, and elsewhere a compression. When $a'/a = \cdot 3486$ these two surfaces coincide, so that the radial strain is nowhere an extension. Lastly, when a'/a is greater than ·3486 the radial strain is everywhere a compression. The radii of the surface or surfaces where the radial strain vanishes are given approximately in the following table for the values 0, ·1, ·2 and ·3 of a'/a:—

TABLE V.

Radii of surfaces over which $\frac{du}{dr} = 0$; $\eta = \cdot 4$.

$a'/a =$	0	·1	·2	·3
$r_1/a =$	$1{\cdot}755a'/a$	·177	·364	·589
$r_2/a =$				·975

§ 14. As illustrating Class IV. we shall consider the limiting value ·5 of η. For it we find approximately

$$a_1'/a = \cdot 4472.$$

Thus when a'/a is less than ·4472 the radial strain vanishes over one surface, $r = r_1$, within the cylinder, being a compression within, an extension outside of this surface. When $a'/a = \cdot 4472$ the radial strain vanishes over the outer surface of the cylinder and elsewhere is a compression. Lastly when a'/a exceeds ·4472 the radial strain is everywhere a compression. The approximate values of r_1/a for the values 0, ·1, ·2, ·3 and ·4 of a'/a are as follows:—

TABLE VI.

Radius of surface over which $\frac{du}{dr} = 0$; $\eta = \cdot 5$.

$a'/a =$	0	·1	·2	·3	·4
$r_1/a =$	$2{\cdot}449a'/a$	·244	·480	·704	·910

§ 15. The expressions for the stresses call for but little remark. In the solid cylinder $\widehat{rr}$ vanishes over the cylindrical surface, and the same is true of $\widehat{\phi\phi}$ when $\eta = \cdot 5$. Elsewhere these stresses are always positive, *i.e.* tensions. The third principal stress, $\widehat{zz}$, vanishes everywhere when $\eta = 0$; but for other values of η it vanishes only over the cylindrical surface

$$r = a/\sqrt{2},$$

being a tension inside, a pressure outside of this surface. It is easily shown that $\widehat{rr}$ is never algebraically greater than $\widehat{\phi\phi}$ nor algebraically less than $\widehat{zz}$. Thus at every point $\widehat{\phi\phi} - \widehat{zz}, \equiv S$, is a

correct measure of the *stress-difference.* Its greatest value, $\bar{S}$, the *maximum stress-difference,* is found in the axis, being given by

$$\bar{S} = \tfrac{3}{8}\,\omega^2\rho a^2\,\{1 - \tfrac{1}{3}\eta/(1-\eta)\} \quad\text{...............(22).}$$

In the hollow cylinder it is obvious from (4) that $\widehat{rr}$ vanishes over both cylindrical surfaces and elsewhere is a tension. Also $\widehat{\phi\phi}$ is always algebraically greater than $\widehat{rr}$ and so is necessarily a tension, though when $\eta = \cdot 5$ it may be vanishingly small. The third principal stress $\widehat{zz}$ is never algebraically greater than $\widehat{\phi\phi}$. It is a tension inside a pressure outside of the surface

$$r^2 = (a^2 + a'^2)/2,$$

where $\widehat{rr}$ is a maximum. The stress-difference may be $\widehat{\phi\phi} - \widehat{rr}$ or $\widehat{\phi\phi} - \widehat{zz}$ according to the axial distance of the point considered, but the maximum stress-difference is always the value of $\widehat{\phi\phi} - \widehat{rr}$ at the inner surface, and is given, since $\widehat{rr}$ is there zero, by

$$\bar{S} = \widehat{\phi\phi}_{r=a'} = \omega^2\rho a^2\,\{3 - 2\eta + (1-2\eta)\,a'^2/a^2\} \div 4\,(1-\eta)\text{...(23).}$$

§ 16. In such a problem as the present the question perhaps of most practical importance is the determination of the greatest safe speed. Under ordinary conditions this is found according to the *stress-difference theory* by attributing to $\bar{S}$ a limit found experimentally; on the *greatest strain theory* $\bar{s}$ is the quantity to which an experimental limit is assigned. I have elsewhere* discussed this question, pointing out that these theories at best can do no more than indicate the limiting stress or strain consistent with the stress-strain relations in the material remaining linear, *i.e.* obeying Hooke's law. There is, however, no mathematical objection to their application to determine a safe working limit, provided this is consistent with the linearity of the stress-strain relations. In the present case the question is complicated by the possibility of the motion becoming unstable. It is obvious, in fact, that in a long thin cylinder there is a danger that the axis under rotation may cease to be straight and may describe a spindle-shaped surface of revolution about the line joining its ends. This has been pointed out by Professor Greenhill†, who has found formulae for the limiting speed, as depending on this kind of instability, in terms of the material and dimensions of the cylinder.

I thus propose in the first place to give tables whence the limiting speeds allowed by the stress-difference and greatest strain theories may be obtained, illustrating their application to special cases; secondly, to give data showing how it may be deter-

* *Phil. Mag.* September, 1891, pp. 239—242.

† *Institution of Mechanical Engineers, Proceedings,* 1883, pp. 182—209, with discussion, pp. 210—225.

mined whether these or the speeds assigned by Professor Greenhill's theory are the lowest; and thirdly, to direct attention to some details of practical interest.

§ 17. All the methods of determining the limiting speed agree in the conclusion that in cylinders of the same material, in which l/a and a'/a have given values, the limiting speed is reached when the velocity ωa at the outer surface attains a certain value. Thus ωa and not ω is the most convenient quantity to tabulate.

On the stress-difference theory we assign to $\bar{S}$ in (22) and (23) a limiting value determined for each material by experiment. The following table is calculated from these formulae, the limiting velocity being termed $\omega_1 a$ for the sake of reference. In using the table $\bar{S}$ has to be replaced by the experimental limit found for the material under consideration, and ρ by its density.

TABLE VII.

Value of $\omega_1 a \div \sqrt{\bar{S}/\rho}$.

η	Solid cylinder	$a'/a = 0$	·2	·4	·6	·8	1·0
0	1·633	1·155	1·147	1·125	1·091	1·048	1·0
·25	1·732	1·095	1·091	1·078	1·058	1·031	1·0
·5	2·0	1·0	1·0	1·0	1·0	1·0	1·0

When less than three decimal figures occur the result is exact.

On the greatest strain theory the limiting speed is found from (12) and (11)—as $\bar{s} = \delta a'/a'$ in the hollow cylinder,—by assigning to $\bar{s}$ an experimental limit. The value of $\bar{s}$ like that of E or ρ varies of course from one material to another. Denoting by $\omega_2 a$ the limiting speed allowed by this theory, we obtain the following results :—

TABLE VIII.

Value of $\omega_2 a \div \sqrt{E\bar{s}/\rho}$.

η	Solid cylinder	$a'/a = 0$	·2	·4	·6	·8	1·0
0	1·633	1·155	1·147	1·125	1·091	1·048	1·0
·25	1·852	1·109	1·104	1·089	1·066	1·036	1·0
·5	2·828	1·069	1·066	1·057	1·043	1·023	1·0

All the results are only approximate except those in the last column.

§ 18. In comparing the results of Tables VII. and VIII. it should be noticed that in a given material $\bar{S} = E\bar{s}$, if the two theories really apply to all forms of strain, because in a bar under simple longitudinal traction, $\bar{s}$ would be the longitudinal strain answering to a traction $\bar{S}$.

It thus appears that the two theories are in exact agreement when $\eta = 0$ for all values of a'/a, and also when $a'/a = 1$ for all values of η. Also while differing in details they both lead to the conclusion that as η increases, other properties of the material being supposed unaltered, the safe speed rises in a solid cylinder but falls in a hollow cylinder for all values of a'/a; and further that in a hollow cylinder of given material as a'/a increases, a remaining constant, there is a steady though not large fall in the safe speed. To this last law the stress-difference theory recognises an exception in the limiting case $\eta = \cdot 5$, when the safe speed is according to it the same for all values of a'/a.

The most striking result is unquestionably the great fall in the safe speed which according to both theories follows the removal of a core however thin it may be, consistent of course with the mathematical conditions for a free surface being satisfied. The magnitude of this fall is the more conspicuous the larger η is. Even for $\eta = 0$ it amounts to over 29 per cent., and for $\eta = \cdot 5$ it amounts on the greatest strain theory to over 62 per cent. For $\eta = \cdot 25$, which is at least a fair approach to what is found in ordinary isotropic materials, the mean of the falls in the safe speeds prescribed by the two theories is approximately $38\frac{1}{2}$ per cent.

While the precise magnitude of the reduction in the safe speed due to the removal of a thin axial core may be questioned by those who regard with distrust existing theories of "rupture", the fact that there is a large reduction must I think be admitted by all who recognise the validity of the present solution, provided the safe speed is really regulated by the elastic state of the material. For our formulae show a large increase in the greatest values of every stress and strain to follow the removal of a thin core; so the material can hardly fail to be brought considerably nearer to that critical condition where the stress-strain relations cease to be appreciably linear, whatever the precise elastic quantity may be on which that condition depends.

§ 19. To illustrate the use of the previous tables, and to give an idea of the range of the numerical magnitudes of the several quantities tabulated, I shall now consider some special cases.

The first case, to which Table IX. refers, is designed to show the magnitudes of the principal displacements and greatest strains when the maximum stress-difference is of given magnitude. The value of η in the material is taken to be $\cdot 25$. Since $(-\delta l/l)$, $\delta a/a$

and $\bar{s}$, when η is given, vary simply as $\bar{S}/E$, it is most convenient to attach a value to this ratio and not to $\bar{S}$ itself. This has the further advantage that $\bar{S}/E$ is a purely numerical quantity, independent of the system of units adopted. It is taken in Table IX. to be ·001. This value is selected principally owing to the facility with which it lends itself to the deduction from the table of numerical values in other special cases. It is not intended to imply that this is the true ratio of the greatest allowable stress-difference to Young's modulus in any actual material.

TABLE IX.

$\bar{S}/E = \cdot 001$; $\eta = \cdot 25$.

	Solid Cylinder	$a'/a=0$	·2	·4	·6	·8	1·0
$(-\delta l/l)\times 10^3=$	·375	·150	·155	·169	·190	·218	·25
$(\delta a/a)\times 10^3=$	·562	·225	·262	·369	·537	·753	1·0
$\bar{s}\times 10^3=$	·875	·975	·976	·980	·985	·992	1·0

The results, with one or two exceptions, are only approximate. In the hollow cylinder $\bar{s}=\delta a'/a'$, so the table gives also the increase in the radius of the inner surface.

§ 20. The velocity ωa, as may be seen by reference to equations (22) and (23) or to Table VII., varies as $\sqrt{\bar{S}/\rho}$. So in calculating it the absolute values of $\bar{S}$ and ρ, or rather their ratio, and not the value of $\bar{S}/E$, is wanted.

Since British engineers seem accustomed to measure velocity in feet per second and stress in tons weight per sq. inch, these units have been adopted in Table X. Two special cases are there dealt with, in both of which η is taken as ·25. The first case, in which the velocity is styled $\omega_1 a$, answers to $\bar{S}=12$ tons wt. per sq. inch, $\rho=7\cdot 5$ times the density of water. This selection is made so as to fit in with the case treated in Table IX. For if, as there, we suppose $\bar{S}/E=\cdot 001$, then $E=12\times 10^3$ tons wt. per sq. inch, i.e. approximately $18\cdot 90\times 10^8$ grammes wt. per sq. cm. Now this value of E and a specific gravity of 7·5 may fairly be taken as representing steel or wrought iron, though rather low values for good material. The second case in Table X., where the velocity is styled $\omega_1'a$, answers to $\bar{S}=1$ ton wt. per sq. inch, $\rho=$ the density of water, or more generally to $\bar{S}=n$ tons wt. per sq. inch, $\rho=n$ times the density of water. This selection is made with a view to facility of application to other special cases. The results are all approximate.

TABLE X.

Velocity in feet per second; $\eta = \cdot 25$.

	Solid Cylinder	$a'/a = 0$	$\cdot 2$	$\cdot 4$	$\cdot 6$	$\cdot 8$	$1\cdot 0$
$\omega_1 a =$	893	565	563	556	546	532	516
$\omega_1' a =$	706	447	445	440	431	421	408

§ 21. The next special case, that treated in Table XI., supposes the greatest strain $\bar{s} = \cdot 001$, while $\eta = \cdot 25$ as before. These are the only data on which $(-\delta l/l)$ and $\delta a/a$ depend; these quantities, when $\bar{s}$ is given, being independent of E or ρ.

TABLE XI.

$s = \cdot 001$; $\eta = \cdot 25$.

	Solid Cylinder	$a'/a = 0$	$\cdot 2$	$\cdot 4$	$\cdot 6$	$\cdot 8$	$1\cdot 0$
$(-\delta l/l) \times 10^3 =$	$\cdot 429$	$\cdot 154$	$\cdot 159$	$\cdot 172$	$\cdot 193$	$\cdot 220$	$\cdot 25$
$(\delta a/a) \times 10^3 =$	$\cdot 643$	$\cdot 231$	$\cdot 268$	$\cdot 377$	$\cdot 545$	$\cdot 759$	$1\cdot 0$

The results with the exception of those in the last column are only approximate.

In calculating S we require in addition to $\bar{s}$ the value of E, but not that of ρ. In Table XII. the value assigned to E is 20×10^8 grammes wt. per sq. cm., which is a fair average for good wrought-iron or steel. For practical convenience the maximum stress-difference is given in tons wt. per sq. inch, taking 70·3083 grammes per sq. cm. as equal to 1 lb. per sq. inch. The results are all approximate.

TABLE XII.

$s = \cdot 001$; $E = 20 \times 10^8$ *grammes wt. per sq. cm.;* $\eta = \cdot 25$.

	Solid Cylinder	$a'/a = 0$	$\cdot 2$	$\cdot 4$	$\cdot 6$	$\cdot 8$	$1\cdot 0$
$\bar{S}$, in tons per sq. inch $=$	14·51	13·025	13·01	12·96	12·89	12·80	12·70

In calculating the velocity we require in addition to the values of $\bar{s}$ and E the value of ρ. In Table XIII. it is taken to be 7·5 times the density of water, while $\bar{s}$ and E have the same values as in Table XII. The velocity is termed $\omega_2 a$ for distinction; it is measured in feet per second. The results are all approximate.

TABLE XIII.

$\bar{s} = {\cdot}001;\ E = 20 \times 10^8;\ \rho = 7{\cdot}5;\ \eta = {\cdot}25.$

	Solid Cylinder	$a'/a = 0$	$\cdot 2$	$\cdot 4$	$\cdot 6$	$\cdot 8$	$1{\cdot}0$
$\omega_2 a$, in feet per second	$= 983$	589	586	578	566	550	531

In comparing the $\omega_2 a$ of Table XIII. with the $\omega_1 a$ of Table X. it must be remembered that the values assigned to E in the two cases are different. The proper basis of comparison may at once be arrived at from the fact that the stress-difference and greatest strain theories are in exact agreement in the limiting case $a'/a = 1$.

§ 22. The results in the last three tables may easily be applied to other special cases when $\eta = {\cdot}25$. Since $(-\delta l/l)$ and $\delta a/a$ vary simply as $\bar{s}$, if any value other than ·001 be assigned to $\bar{s}$ we have only to alter the numbers in Table XI. in the same proportion. We may also make use of the facts that S varies directly as $E\bar{s}$, and $\omega_2 a$ varies as $\sqrt{E\bar{s}/\rho}$, to adapt Tables XII. and XIII. to other cases.

For instance, let us take flint-glass, assuming for it $\eta = {\cdot}25$ and allowing to $\bar{s}$ the value ·0008. Let us take $E = 6 \times 10^8$ grammes wt. per sq. cm., and $\rho = 2{\cdot}94$ times the density of water, values which are approximately the mean of those given by Professor Everett*. Then to obtain numerical results for this case we have only to multiply the values of $(-\delta l/l)$ and $\delta a/a$ in Table XI. by ·8, the values of $\bar{S}$ in Table XII. by $(6/20) \times {\cdot}8$ or ·24, and the values of $\omega_2 a$ in Table XIII. by $\sqrt{(6/20) \times {\cdot}8 \times (7{\cdot}5/2{\cdot}94)}$ or ·7825 approximately.

§ 23. We have next to consider the limits which Professor Greenhill has found for the safe speed when none but "centrifugal" forces are supposed to act. He imagines the cylinder bent under rotation so as to be in equilibrium under the "centrifugal" forces, which tend to increase its bending, and the elastic forces whose tendency is to keep it straight. He thence arrives at a relation, varying with the terminal conditions, between the velocity of rotation and what is practically the ratio of the length to the diameter of the cylinder. Taking the velocity and cross section as given, Professor Greenhill regards this as fixing the greatest length of cylinder in which rotation is stable.

* *Units and Physical Constants*, 2nd Edition, Art. 64.

The method is analogous to that whereby we obtain Euler's formula for the greatest length permissible in a strut of given section subjected to a given load. If instead of fixing the velocity of the cylinder we suppose the ratio of length to diameter given us along with the cross section, Professor Greenhill's relation assigns a limiting speed which cannot be exceeded without exposing the cylinder to buckling.

The physical problem treated by Professor Greenhill is one of great difficulty, and doubts may be entertained as to how closely it is reproduced in the mathematical problem which he has solved. This question would however lead us too far afield, and we shall here consider merely the conclusions to which the instability formulae lead, throwing all responsibility for the exactness of the instability theory upon its author.

§ 24. For the case of simple rotation Professor Greenhill gives two formulae*. In deducing the first he takes

$$y = 0 = \frac{dy}{dx}$$

at the ends of the cylinder, where y denotes the perpendicular from a point on the axis, supposed bent under rotation, on the straight line coinciding with its undisturbed position. This supposes the axis constrained to retain its original direction at the ends. The terminal conditions assumed in the second formula are

$$y = 0 = \frac{d^2y}{dx^2}.$$

This answers to zero curvature of the axis at the ends, or on Professor Greenhill's interpretation of the problem to the vanishing of the elastic bending couple.

The conclusions to which the formulae lead appear at first sight widely different, the length allowed by the first set of terminal conditions in a cylinder of given section rotating with a given speed being according to Professor Greenhill $9{\cdot}46/\pi$ times—or approximately thrice—that allowed by the second. The difference is however in considerable part due to a slight slip made by Professor Greenhill in attaching a numerical value to a quantity he terms $\frac{1}{2}\mu l$. For this he quotes (*l.c.* p. 199) the value 4·73, whereas in reality

$$\tfrac{1}{2}\mu l = 2{\cdot}36502,$$

or only half this value.

* Those numbered (19) and (20) in Professor Greenhill's paper, pp. 199 and 200.

Amending this numerical coefficient, and employing as previously $2l$ for the length of the cylinder, in place of Professor Greenhill's l, we may represent his two results under the forms

$$\omega^2\rho/E\kappa^2 = (2\cdot36502/l)^4 \quad\ldots\ldots\ldots\ldots\ldots\ldots(24),$$
$$\omega^2\rho/E\kappa^2 = (\pi/2l)^4 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25),$$

where κ denotes the radius of gyration of the cross section about a perpendicular through its centre to the plane of bending. The rest of the notation has its previous signification. Professor Greenhill apparently introduces no restriction as to the nature of the cross section; thus the formulae apply presumably to hollow as well as solid cylinders.

The terminal conditions assumed in (25) are based on the elastic theory adopted by Professor Greenhill; while (24) merely assumes the direction of the axis fixed at the ends, and so seems exposed to fewer uncertainties. I shall thus direct my attention to the first of the two formulae; but results answering to the second can be easily derived from those answering to the first by replacing the factor $(1\cdot18251)^4$ in equations (26) to (29) below by $(\pi/4)^4$.

§ 25. Let us then suppose that in a certain isotropic solid circular cylinder the limiting velocity as prescribed by (24) is such as to cause in the material a maximum stress-difference $\bar{S}$. Then remembering that $a^2/4$ is the value of κ^2 in a circle, and that the angular velocity which appears in (24) is the same as appears in (22), we immediately deduce

$$\bar{S}/E = (1\cdot18251)^4 (a/l)^4 (3-4\eta) \div 2(1-\eta) \ldots\ldots(26).$$

Similarly if $\bar{s}$ be the greatest strain answering to the velocity prescribed by (24) we find by means of (12)

$$\bar{s} = (1\cdot18251)^4 (a/l)^4 (3-5\eta) \div 2(1-\eta) \ldots\ldots(27).$$

In a circular annulus $\kappa^2 = (a^2 + a'^2)/4$. Thus, remembering that $\bar{s} = \delta a'/a'$, we find from (23) and (11) for the maximum stress-difference and greatest strain in a hollow cylinder, answering to the limiting velocity prescribed by (24), the respective values:

$$\bar{S}/E = (1\cdot18251)^4 (a/l)^4 (1 + a'^2/a^2) \times \{3 - 2\eta + (1-2\eta)\, a'^2/a^2\} \div (1-\eta) \ldots\ldots\ldots\ldots\ldots(28),$$

$$\bar{s} = (1\cdot18251)^4 (a/l)^4 (1 + a'^2/a^2) \{3 + \eta + (1-\eta)\, a'^2/a^2\} \ldots(29).$$

§ 26. It is obvious that the maximum stress-difference and greatest strain answering to the limiting velocity prescribed by the instability formula always diminish very rapidly as the ratio of the length of the cylinder to its diameter increases. Also when

the magnitude of $\bar{S}/E$ or $\bar{s}$ is known for any one value of a/l, it is a simple matter to obtain its magnitude for any other value of a/l supposing η and a'/a unaltered. The following table gives the approximate numerical magnitudes of $\bar{S}/E$ and $\bar{s}$ when $l = 10a$, the value of η in the material being ·25.

TABLE XIV.

Solid cylinder	a'/a 0	·2	·4	·6	·8	1·0
$(\bar{S}/E) \times 10^3 =$ ·261	·652	·683	·780	·950	1·206	1·564
$\bar{s} \times 10^3 =$ ·228	·635	·667	·764	·936	1·196	1·564

§ 27. By assigning a given value to $\bar{S}/E$ or $\bar{s}$ in equations (26) to (29) we obtain a value of l/a which gives the shortest cylinder in which so large a value of $\bar{S}/E$ or $\bar{s}$ would be reached when the velocity was kept within the range prescribed by the Greenhill formula (24). If this value of $\bar{S}/E$ or $\bar{s}$ be the extreme limit of safe working in the material considered, the corresponding value of l/a determines the shortest cylinder in which the safe speed can properly be assigned by (24). Suppose for instance $\eta = 0$, the case in which the stress-difference and greatest strain theories agree, and let $\bar{S}/E = \bar{s} =$ ·001 denote the safe working limit, then the values of l/a for the shortest cylinders in which it is legitimate to determine the safe speed by means of (24) are those given in the following table:—

TABLE XV.

Solid cylinder	a'/a 0	·2	·4	·6	·8	1·0
$l/a =$ 7·36	8·75	8·87	9·20	9·72	10·39	11·18

For the limiting case $a'/a = 1$ the result is independent of η, and in fact the value of η is of comparatively little importance except in the case of the solid cylinder or when a'/a is small. The difference between the limiting values of l/a allowed to the application of (24) in a solid cylinder and in a hollow cylinder in which a'/a is vanishingly small is least when $\eta = 0$. It is greatest when $\eta =$ ·5, in which case we find for l/a in a solid cylinder answering to $\bar{S}/E =$ ·001, and to $\bar{s} =$ ·001 the respective values 6·65 and 5·59; while for the hollow cylinder $a'/a = 0$ the corresponding values are no less than 9·40 and 9·10.

Since l/a according to the formulae (26) to (29) varies as $(E/\bar{S})^{\frac{1}{4}}$ or $(1/\bar{s})^{\frac{1}{4}}$ it is easy to adapt results such as those of Table XV. to cases when values other than ·001 are allowed to $\bar{S}/E$ or $\bar{s}$. A caution may however be not unnecessary as to the use of such results, viz. that unless η vanish or be very small the solution obtained in the present paper is not altogether trustworthy when l/a is markedly less than 10, and that the Greenhill theory is probably in such a case still more untrustworthy.

§ 28. Tables XIV. and XV. show that Professor Greenhill's formula (24), assuming his theory trustworthy in itself, ought not to be applied in determining the limiting speeds in hollow cylinders whose length is less than 12 or 13 times their diameter, without a check being applied by reference to the results of the present paper. The thinner the walls of the cylinder the more necessary is the check. In solid cylinders, a check of this sort is less necessary, but it must be remembered that when l/a is less than 10 the hypotheses on which (24) is based can hardly be considered satisfactory.

§ 29. While the exact law laid down by the Greenhill formula, viz. that the limiting speed ωa as far as instability is concerned varies as $(a/l)^2$, cannot be relied on in short cylinders, there can be little doubt that there is a continuous and rapid diminution in the tendency to instability as l/a is reduced below 10. On the other hand, a large increase in the limiting speed allowed by the stress-difference and greatest strain theories can hardly accompany this reduction of l/a.

For while in short cylinders the strains and stresses will doubtless for ordinary values of η vary appreciably with z, the mean value of a certain strain or stress-difference for a given value of r when taken between $\pm l$ seems hardly likely to vary much with l/a.

Thus it is *a priori* improbable that the greatest values reached by these quantities in a short cylinder can be much less than the values attained in a long cylinder, the values of ωa and a'/a being the same, though conceivably in some cases they may be appreciably greater (cf. § 31).

The following considerations afford strong support to this view: The expressions we have found for the maximum stress-difference and greatest strain both in thin disks and long cylinders, when ωa, a'/a, ρ and E are treated as constants, vary with the value of η only within comparatively narrow limits (see § 17). Unless then the influence of η on the magnitude of these quantities be very much greater in cylinders of intermediate

length, a fair idea of this magnitude in any material may be derived by supposing η to vanish. But our solution is exact when $\eta = 0$, and the maximum stress-difference and greatest strain are then wholly independent of l/a.

There is thus, on various grounds, a strong presumption that in any isotropic material the limiting safe speeds allowed by a complete elastic theory in short cylinders would not differ very much from those which the elastic theory of the present paper allows in long cylinders. Thus, even if Professor Greenhill's method could be trusted when l/a ceases to be large, recourse would require to be had to the strict elastic theory and not to his formulae to fix a limiting speed.

§ 30. In Professor Greenhill's paper, and in the discussion which follows it, reference is made to propeller shafts in steamers. These shafts, at least in large steamers, are of a length very great compared to their diameter. The "Servia" for instance is said to have a solid shaft 164 feet long and only $22\frac{1}{2}$ inches in diameter. It seems however the practice to support the shaft at intervals; in the "Servia" for example there are bearings making of the shaft 8 lengths. In such a case the distance between the bearings ought presumably to be regarded as the length $2l$ in applying the instability formula. The value of l/a may perhaps still be great enough for a legitimate application of the Greenhill theory, but it is by no means so great that a consideration of the magnitude of the elastic strains and stresses can be safely dispensed with, at least in wrought-iron. When a shaft of this kind is hollow, as is sometimes the case, the magnitudes of the elastic strains and stresses given by our formulae should be looked to, even if the material be the best steel. These remarks assume of course that none but "centrifugal" forces act. In propeller shafts under normal conditions this is far from true, there being torsional and longitudinal forces as well, whose effects may be large compared to those of the "centrifugal" forces. When such a compound system of forces acts, the limiting speed according to the elastic theory is to be got by superposing the displacements arising from the several sources, and then ascribing limiting values to the maximum stress-difference or greatest strain given by the complete solution. It would however be straying too far from our main subject to discuss this matter further.

§ 31. Reference has already been made to the comparatively small difference between the limiting speeds allowed by the elastic theories of "rupture" in long cylinders and in thin disks. The exact relations between these speeds are as follows:

Let the limiting speeds in a long solid cylinder and in a complete disk, both of radius a and of the same isotropic material, be denoted by ω_1 and Ω_1 respectively on the stress-difference theory, by ω_2 and Ω_2 respectively on the greatest strain theory. Then by reference to my previous paper, we easily find

$$\omega_1^2/\Omega_1^2 = 1 + \eta\ (2 - \eta)/(3 - 4\eta) \quad \text{..............(30)},$$
$$\omega_2^2/\Omega_2^2 = 1 + \eta^2\,(1 + \eta)/(3 - 5\eta) \quad \text{..............(31)}.$$

Hence ω_1/Ω_1 and ω_2/Ω_2 both vary from 1 when $\eta = 0$ to $\sqrt{7/4}$, or 1·3229 approximately, when $\eta = \cdot 5$. For $\eta = \cdot 25$ we have approximately

$$\omega_1/\Omega_1 = 1\cdot 104, \quad \omega_2/\Omega_2 = 1\cdot 022.$$

For a hollow cylinder and an annular disk of the same isotropic material, with the same values of a and a', we find, distinguishing the results from those given above by dashed letters:

$$(\omega_1'/\Omega_1')^2 = 1 - \eta^2\,(1 - a'^2/a^2) \div \{3 - 2\eta + (1 - 2\eta)\,a'^2/a^2\} \text{...(32)},$$
$$(\omega_2'/\Omega_2')^2 = 1, \text{ for all values of } \eta \text{ and of } a'/a \text{.................(33)}.$$

Thus the limiting speeds on the greatest strain theory are here identical, and on the stress-difference theory the difference between the speeds is extremely small for ordinary materials, especially when $1 - a'/a$ is small. It is worth noticing that according to the stress-difference theory the limiting speed is less in the long hollow cylinder than in the corresponding thin disk, whereas it is greater according to both theories in the long solid cylinder than in the complete disk of the same radius.

Monday, Feb. 22, 1892.

PROFESSOR G. H. DARWIN, PRESIDENT, IN THE CHAIR.

The following communications were made to the Society:

(1) *Some preliminary notes on the anatomy and habits of* Alcyonium digitatum. By SYDNEY J. HICKSON, M.A., Fellow of Downing College, Cambridge.

Alcyonium digitatum is one of the most difficult Coelenterates to kill in a fully expanded condition. In the first place it is only

extremely rarely that the large proportion of the polypes of a specimen in an aquarium fully expand themselves, and when they are in that condition the slightest touch or irritation of any part of the colony causes an immediate contraction of the tentacles. Again when a favourable opportunity arises it is found that all the neutral killing reagents such as corrosive sublimate, etc. fail to kill the polypes before they have time to partially retract. The only method that gives tolerably satisfactory results is Lo. Biancho's No. II. Chromo-acetic acid method, and this of course partially dissolves the calcareous spicules.

When a living specimen of an Alcyonium digitatum is examined in an aquarium the polypes may frequently be observed in various stages of expansion and retraction. Sometimes all the polypes are completely retracted, but I have never yet observed in any specimen all the polypes fully expanded at the same time. By far the most frequent condition of the Alcyonium is one in which a few polypes here and there are fully expanded, others expanded but with their tentacles contracted, and others only just protruding from the surface of the colony.

These two stages are the normal ones that each polype passes through in reaching complete retraction from complete expansion. When the polype is completely expanded both the body-wall and the tentacles are delicate and transparent.

The first stage in the retraction is the contraction of the tentacles. The crown of the polype becomes roughly octagonal in shape with an obtuse solid knob—the contracted tentacle at each angle.

In the next stage the contracted tentacles bend over towards the mouth and concurrently with the retraction of the body of the polype they sink into a circular fold of the body-wall.

The invagination of the polype then proceeds at the base until the crown covered by the fold of body-wall sinks below the surface of the colony.

When the crown has sunk below the surface of the colony the aperture is closed by the folding of the delicate body-wall of the base of the polype over the crown, but when the colony enters into a state of complete contraction, as it does for example when it is taken out of the water for a few minutes, the tough obtuse surface of the colony contracts over this delicate base leaving only a star-like slit to mark the position of the retracted polype.

The ectoderm is in the uninjured specimens composed of several layers of cells. Under unhealthy conditions the superficial layers are apt to slough.

The stomodaeum of each fully-developed polype opens into a long coelenteron that passes down to the base of the colony. In a longitudinal section through a branch of the colony the

coelentera spread from the base in a fan-like manner towards the periphery.

Between the coelentera there is a dense clear mesogloea containing spicules, endodermic cell cords, a very delicate network of nerve (?) fibrils and cells.

The young polypes originate in these endodermic cords, the coelentera being developed later, and do not communicate directly with the coelentera of the older polypes until they are nearly full grown.

The endodermic cords are usually described as canals but there is no evidence of the presence of a lumen. Numerous injection experiments failed to prove the existence of any cavity in these structures.

The nerve (?) netwoık can only be made out in fresh osmic acid preparations. It consists of a fine network of delicate fibrils connecting isolated mono- bi- and tripolar cells. It is difficult to trace in the peripheral parts of the colony, since the presence of a very large number of spicules makes it a matter of some difficulty to cut very thin sections of the fresh tissues.

When the tide is low in the tropics some forms of Zoophytes, such as Tubipora, Clavularia, Sarcophytum, the Astraeidæ and a few others, remain expanded until the water actually leaves the edge of the reef. Others on the other hand, such as Heliopora, Millepora and most of the Madrepores, completely retract while there is still a foot or more of water covering them. Some Zoophytes in fact appear to anticipate low tide before others, and it occurred to me that this might be to a certain extent due to the development of a rhythm similar to the rhythmic movements of certain plants.

In order to determine if possible the truth or falsity of this supposition I made last autumn a series of experiments upon Alcyonium digitatum in the tanks at the Plymouth laboratory.

I placed a number of specimens of Alcyonium digitatum, collected partly in the shallow water of the Catwater and partly in deeper water off the Eddystone lighthouse, in one of the tanks, and I noticed that nearly all of them contracted completely twice in every twenty-four hours for the first three or four days.

Those that did not contract in this manner soon showed bubbles of gas in their tissues and commenced to putrify.

I also placed a number of specimens in another tank in which I arranged an artificial tide by means of siphons. It was so arranged that the water should run off—but not so completely as to leave the Alcyoniums uncovered—in twelve hours and fill up again in twelve hours. The Alcyoniums contracted with tolerable regularity twice in twenty-four hours for the first two days and then contracted quite irregularly, some only once, some twice. At the end of a fortnight two out of the three that remained in

a healthy condition contracted regularly only once in twenty-four hours.

These experiments appear to indicate 1st that there is a rhythmic contraction of the polypes of Alcyonium digitatum in the normal conditions twice in every twenty-four hours, 2nd that a new rhythm may be induced by an artificial tide of different duration to the natural one.

On the other hand the results of the investigation are not altogether satisfactory on account of 1st the large number of specimens that became unhealthy during the progress of the experiment, 2nd the fact that the only three specimens that survived in the tank with the artificial tide were taken from the deep water off the Eddystone.

Far more satisfactory results would probably be obtained if experiments were tried upon a number of Alcyoniums taken from shallow water direct and placed in the tanks. Many of the specimens collected from the Catwater were probably taken by the trawlers from deep water and thrown overboard as they entered the harbour and were in consequence in an unhealthy condition when collected.

The subject however seems to me to be worthy of further investigation.

(2) *On the action of Lymph in producing Intravascular Clotting.* By LEWIS SHORE, M.D., Fellow of St John's College.

From 1883 to 1889 numerous papers[1] were published by Wooldridge on intravascular clotting. He showed that when a watery extract of lymphatic glands, of testis, of thymus and of other tissues was injected into the blood, death by more or less extensive intravascular clotting was produced. He further showed that this property depended on the presence in the extracts of substances, proteid in nature, named by him tissue-fibrinogens. He could also obtain these bodies from chyle[2], but he made no definite statement that ordinary lymph could produce intravascular clotting; but it is clear from his paper on Auto-infection in Cardiac disease[3] that he attributed to it such power, although he had no definite experimental proof to bring forward. When we consider that he proved the power to reside in the juice expressed from lymphatic glands it seems very remarkable that he did not try lymph drawn from the thoracic duct. The various

[1] Chiefly in *Proceedings of Royal Soc.*, and *Journal of Physiology*.
[2] Ludwig's *Festschrift*, 1887. 221.
[3] *Proceedings of Royal Society*, XLV. 309.

extracts and fluids he injected are all artificial, and none, not even the expressed juice from the lymphatic glands, can be considered as normally present in the body in exactly the form in which he injected them. It was therefore of the greatest importance from his point of view to show that the lymph as it is normally passing to the blood carries with it the very substances which the artificial extracts contain, viz. tissue fibrinogens, and that the lymph itself when injected in the same way as the extracts, does by virtue of these bodies produce intravascular clotting. The present paper is a preliminary account of a few experiments I have so far made which supplement and continue Wooldridge's work on this point. The observations were made during the continuation of my work on the action of peptone on the clotting of Lymph and Blood[1]. It was shown by Heidenhain[2] that when peptone is injected into the blood, the flow of lymph is much increased and the percentage of proteids in it increased. It was possible then, that the blood had lost to the lymph certain proteids necessary to clotting, and that the cause of the loss of clotting power was to be sought by studying the relations of the lymph to the blood. The addition of peptone lymph to peptone blood out of the body did not lead to clotting. I then turned to the intravascular injection of lymph, and first of all to normal lymph.

The lymph was collected from the thoracic duct of a dog in the ordinary way. In the first experiment 12 c.c. of lymph could be collected before it clotted. This was quickly injected into the jugular vein of a rabbit, there was some rapid respiration, and in 2—3 minutes the animal was dead. The heart, and all the arteries and veins, even their small branches, as far as traced, contained only clotted blood. The clotting was complete so that long "blood casts" could be drawn from the vessels. A smaller quantity of the same lymph, 5 c.c., was then collected and injected into another rabbit. In about one minute the animal was dead, and general intravascular clotting was found as before. It was at once considered possible that the difference in the species of the two animals might be of great importance. Richet and Hericourt[3] found that 12 c.c. of defibrinated dog's blood was sufficient to kill a rabbit, but the condition found after death was not one of general intravascular clotting, as only a few small clots were formed, although there was a profound influence on the red corpuscles. I injected 15 c.c. of freshly drawn arterial blood from the same dog without producing any effect on the clotting power of the rabbit's blood, neither hastening or retarding clotting. Lymph from the same dog was allowed to clot, and 15 c.c. of the lymph serum, injected into a rabbit, also produced no effect.

[1] *Journal of Physiology*, XI. 561.
[2] Pflüger's *Archiv*, XLIX. 209.
[3] *Comptes rendus de la Soc. de Biol.* CVII. 748 and CX. 1282.

Subsequent experiments showed that the intravascular clotting produced in the rabbit by the injection of the lymph of the dog is not always so complete as in the experiments just recorded. In many cases the animal died very soon after the injection was completed, but on examination clots were found only in the heart. In some cases the clots were restricted to the right side of the heart. Again in some cases the animals died within two or three minutes after the injection, and generally after some disturbed respiration, but the blood even in the heart was found to be fluid. Sometimes however in these cases a careful examination has revealed minute clots in the right side of the heart and in the pulmonary vessels. The blood in these cases usually clotted almost instantly it was shed. Several cases have occurred in which the injection of lymph produced no obvious effect on the rabbit.

I am not able at present to fully explain this considerable variability in action. The explanation is to be sought rather in the condition of the lymph injected than in variations in the blood of the rabbit. It would be naturally expected that the variation, in the density, the amount of proteids, and the number of leucocytes, &c. which occur in the lymph of the dog, would influence the amount necessary to be injected to produce intravascular clotting. The rate of clotting of the lymph limits the amount which can be used, and it is rare that more than 10 c.c. of normal lymph can be collected before clotting has set in. This is a serious difficulty in the work, and in the negative cases observed a rapid clotting lymph was generally recorded. The most active lymph in producing intravascular clotting is that obtained from a dog in active digestion. Wooldridge also observed that the intravascular clotting produced by the injection of tissue-fibrinogen is more readily produced and is more extensive when the animal experimented with is in full digestion. I have however observed that the opaque white lymph in these cases is sometimes inert.

Large quantities of lymph, which does not clot, may be obtained by the injection of peptone into the vascular system. Such lymph is however unable to cause intravascular clotting when injected into a rabbit. So also is "salted lymph," that is lymph received into solutions of neutral salts. That this can be explained by the presence in the lymph in these cases of substances opposing the action of the clotting-exciting substance, is probably shown by the fact that tissue-fibrinogen can be separated from the plasma of such lymph, and that its injection leads to intravascular clotting.

Tissue-fibrinogen was obtained from lymph in the following way. The lymph was allowed to drop directly from the thoracic duct into a very small quantity of very dilute acetic acid, a few more drops of the diluted acid being added from time to time as

the amount collected increased. A precipitate was thus at once formed, this was separated by centrifugalisation, and then twice washed with water as Wooldridge directed. On the addition of a small quantity of alkali, very dilute sodic hydrate, the precipitate almost completely dissolves. (It consists partly of leucocytes separated by the centrifuge.)

The tissue-fibrinogen solution itself clotted in 2—5 minutes to a firm jelly. If injected at once into a rabbit, the animal dies almost immediately, and clots are found in the heart and in some cases the portal system. This result was however not always produced, sometimes the animal remained alive, but its blood was found to have lost its clotting power. This is in full accord with the observations of Wooldridge, with tissue-fibrinogen obtained from glands and testis.

A substance, which produces intravascular clotting, may be obtained from lymph by a method indicated by Alexander Schmidt[1]. He found that an alcoholic extract of lymphatic glands, of liver, and other tissues contained a substance which excited clotting in horse's plasma. I allowed lymph to drop from the thoracic duct directly into alcohol, the precipitate was then shaken with the alcohol for several hours, and then the alcohol was poured off and evaporated to dryness at 37°, and the residue shaken up with a little water. A considerable portion dissolved, and 4 c.c. of this, which formed a clear, slightly pinkish coloured solution, was injected into the jugular vein of a rabbit. The animal died at once, and the heart and large veins were full of clotted blood. The alcoholic extract of blood treated in the same way does not produce intravascular clotting.

(3) *On the fever produced by the Injection of sterilized* Vibrio Metschnikovi *cultures into rabbits.* By E. H. HANKIN, B.A., Fellow of St John's College, and A. A. Kanthack (M.B. Lond.), St John's College.

It is well known that a fever can be produced in rabbits by intravenous injection of the products of the growth of Vibrio Metschnikovi, a microbe closely allied to the Cholera-germ. The serum of rabbits possesses a certain power of killing the Vibrio Metschnikovi, and we have attempted to find out whether this 'bactericidal power' is increased during or after the fever produced by the method above mentioned. We have also noted the numbers of leucocytes present in the rabbit's blood before and after the fever, in the hope of being able to establish some relation between

[1] *Centralblatt für Physiologie*, IV. 257.

the numbers of leucocytes present in the blood, and the degree of 'bactericidal power' possessed by the serum derived from it.

The following is a short summary of our observations:

(1) *Methods employed.* The microbe was grown in calves-foot bouillon for fourteen days. The culture fluid was then sterilized by heating in an autoclave to 115° C. for a quarter of an hour. Of the turbid liquid thus obtained $\frac{1}{4}$ to $\frac{3}{4}$ cc. was injected into the lateral ear vein of a rabbit by means of a Koch's syringe. The temperature of the rabbit was taken in the usual way every half-hour, and when desired the animal was killed by cutting the carotid artery under antiseptic precautions. The blood was received into a sterilized centrifuge tube and was centrifugalised as soon as clotting had taken place. By this means a large quantity of serum can be obtained within a few minutes of the death of the animal.

(2) *The rise of temperature.* Almost always such an injection was followed by a rise of temperature of 1 to $2\frac{1}{2}$ degrees Centigrade. A temperature of 41·4° C. was the highest that we have hitherto observed. The temperature generally rises quickly and within an hour after the injection may be two degrees above the normal. The maximum temperature is reached in $1\frac{1}{2}$ to $2\frac{1}{2}$ hours after injection. The fall is far more gradual, but the temperature has generally returned to the normal within six hours of the injection.

(3) *Effect of the injection on the number of leucocytes present in the blood.* Soon after the injection the number of leucocytes present falls to a fraction of the normal. Within an hour of the injection the larger leucocytes with lobed nuclei may have diminished to so great a degree that several preparations have to be examined before one of them can be seen. The smaller white blood corpuscles however, consisting of one spherical deeply-staining nucleus and a trace of protoplasm (= lymphocyte), are not affected to so great an extent. Often scarcely any decrease in their number can be observed. At about the period of the acme of the fever a slight increase in the number of leucocytes is generally to be seen, so that almost as many are present as in the blood of a normal rabbit. At a later stage, when the fever has nearly passed off, a great increase in the number of leucocytes is always met with. In the majority of cases this increase is quite sudden. In one case for instance a preparation of blood taken $4\frac{1}{2}$ hours after injection showed scarcely any of the larger leucocytes. A similar preparation made $4\frac{1}{2}$ hours after injection, showed a well-marked leucocytosis. Five hours after the injection so many leucocytes were present that 40 to 50 could often be counted under the microscope in a single field of view. The increase affects, for the most part, the larger leucocytes with lobed nuclei. As a

rule only a slight increase in the number of lymphocytes present can be observed. After two days the blood still exhibits the same increased number of leucocytes. At the end of a week a certain amount of leucocytosis is still present, though the excess appears, in some of the few cases examined at this period, to consist to a great extent of the smaller lymphocytes.

(4) *Changes in the bactericidal power of the serum.* Blood taken from a rabbit during the diminution stage of the number of leucocytes present, yields serum having less power of killing the Vibrio Metschnikovi than is normal. Blood taken as soon as leucocytosis has appeared yields serum which exerts a bactericidal action on this microbe which may be two or three times as great as that possessed by serum from an ordinary rabbit. It is noteworthy that this increase is *not* proportional to the increase in the number of leucocytes present. Blood taken twenty-four to forty-eight hours after injection yields serum which possesses a bactericidal power for Vibrio Metschnikovi far above the normal. In one experiment a given volume of serum was found to be able to kill about 60 times as many microbes as could be killed by the same quantity of normal serum.

(4) *Note on the Method of Fertilisation in* Ixora. By J. C. Willis, B A., "Frank Smart" Student in Botany, Gonville and Caius College.

In *Ixora salicifolia*, D.C., the flowers are massed together in large corymbs, and thus rendered very conspicuous. A ring-shaped nectary upon the epigynous disc secretes honey, which is protected from rain and from short-lipped insects by the corolla-tube, whose length is 30 mm. and diameter about 1·5 mm. The anthers dehisce in the bud, introrsely, shedding their pollen upon the style, whose stigmas are tightly closed together, and thus protected from the pollen. When the flower opens, the stamens bend outwards and downwards until the anthers are below the rim of the corolla, when they usually fall off altogether. The style presents the pollen to insects visiting the flower. After the lapse of about twenty-four hours the stigmas separate from one another, and the flower is now in its second stage. The stigmas do not roll back so far as to effect autogamy.

A similar mechanism appears to occur in *I. coccinea*, and, so far as can be judged from dried specimens, in many other species of the genus.

Monday, March 7, 1892.

PROFESSOR G. H. DARWIN, PRESIDENT, IN THE CHAIR.

The following communications were made to the Society:

(1) *Some Experiments on Electric Discharge.* By Professor THOMSON.

A series of experiments were shown in which the electric discharge took place in bulbs without electrodes. It was shown that the colour of the discharge through the same gas varied very greatly with the density of the gas and the intensity of the discharge. This was illustrated by two bulbs each containing air; the discharge through one was a bright blue and through the other an apple-green. Another experiment showed that gas at a very low pressure could not act as an electromagnetic screen, though it did so at a high pressure. The laws governing the absorption of energy by conductors placed near very rapidly alternating currents were illustrated by experiments which showed that there was much greater absorption of energy by small pieces of tin-foil than large masses of brass or copper.

(2) *On the perturbation of a comet in the neighbourhood of a planet.* By G. H. DARWIN, F.R.S., Plumian Professor and Fellow of Trinity College.

In Chapter II. of Book IX. of the *Mécanique Céleste*, Laplace considers the transformation of the orbit of a comet when it passes a large planet. His object is to show that the action of Jupiter suffices to account for the disappearance of Lexell's comet after 1779.

He remarks that if a comet passes very near to Jupiter, it will throughout a small portion of its orbit move round the planet almost as though it were unperturbed by the Sun, and that both before its approach to and after its recession from the planet it will move round the Sun almost as though it were unperturbed by the planet. The nature of the orbit of the comet will usually be much transformed by its encounter with the planet. It is clear then that there must be some surface surrounding the planet which separates the region, inside of which the comet moves nearly round the planet, from the region in which it moves nearly round the Sun. Such a surface is to be found by the comparison of the ratio of the perturbing force

to the central force in the motion round the Sun with its value in the motion round the planet. There is a certain surface at which this ratio will be the same in the two cases, and this is the surface required for the proposed approximate treatment of the problem.

Now it does not appear to me that Laplace makes any attempt to show that such a surface is even approximately spherical, but he assumes that what has been called "the sphere of Jupiter's activity" is a true sphere, and determines its radius by the consideration of a special case.

The object of the present note is then to treat this problem more fully than does Laplace, and to investigate the nature of the surface in question.

It will appear that whilst Laplace's result is accurate enough for the purpose for which it is intended, yet a slightly different value for the radius of the sphere of activity would be more nearly correct.

Let R, r be the radii vectores of Jupiter and of the comet, and let ρ be the distance of the comet from Jupiter.

Let ω be the angle between R and r, and θ the angle between R produced and ρ.

Let S, M, m be the masses of the Sun, Jupiter and the comet.

Let P, T be the disturbing forces along and perpendicular to ρ, which act on the comet in its motion round the Sun; let F be the resultant of P, T; and let C be the central force acting on the comet.

Let $\mathfrak{P}$, $\mathfrak{T}$, $\mathfrak{F}$, $\mathfrak{C}$ be the similar things, also with reference to ρ, in the motion of the comet round Jupiter.

Now we want to find a surface with reference to Jupiter such that outside of it the comet moves approximately in a conic section round the Sun, and inside of it in a conic section round Jupiter.

If we consider a surface such that

$$\frac{F}{C} = \frac{\mathfrak{F}}{\mathfrak{C}},$$

we shall have what is required.

By the ordinary theory the disturbing function for the motion of the comet round the Sun as perturbed by Jupiter, is

$$M\left\{\frac{1}{\rho} - \frac{r}{R^2}\cos\omega\right\}.$$

But $r\cos\omega = \rho\cos\theta + R$, hence the disturbing function is

$$M\left\{\frac{1}{\rho} - \frac{1}{R} - \frac{\rho}{R^2}\cos\theta\right\}.$$

Differentiating with respect to ρ and θ, we have

$$P = -M\left\{\frac{1}{\rho^2} + \frac{1}{R^2}\cos\theta\right\},$$

$$T = \frac{M}{R^2}\sin\theta.$$

Hence
$$F^2 = M^2\left\{\frac{1}{\rho^4} + \frac{1}{R^4} + \frac{2}{\rho^2 R^2}\cos\theta\right\}$$
$$= \frac{M^2}{\rho^4}\left\{1 + 2\frac{\rho^2}{R^2}\cos\theta + \frac{\rho^4}{R^4}\right\}.$$

But
$$C = \frac{S+m}{r^2},$$

and thus
$$\frac{F^2}{C^2} = \frac{M^2}{(S+m)^2}\frac{r^4}{\rho^4}\left\{1 + 2\frac{\rho^2}{R^2}\cos\theta + \frac{\rho^4}{R^4}\right\}.$$

Again the disturbing function for the motion of the comet round Jupiter as perturbed by the Sun is

$$S\left\{\frac{1}{r} + \frac{\rho}{R^2}\cos\theta\right\}.$$

It will be seen that the sign of the second term is here $+$, because the angle between R and ρ is $\pi - \theta$.

In this formula we have

$$r^2 = \rho^2 + R^2 + 2\rho R\cos\theta.$$

Hence differentiating with respect to ρ and θ, we have

$$\mathfrak{P} = -S\left\{\frac{\rho}{r^3} + R\left(\frac{1}{r^3} - \frac{1}{R^3}\right)\cos\theta\right\},$$

$$\mathfrak{T} = S \,.\, R\left(\frac{1}{r^3} - \frac{1}{R^3}\right)\sin\theta.$$

Now we might proceed to square these two and add them together to find $\mathfrak{F}^2$, and so go on to find the rigorous expression for $\mathfrak{F}/\mathfrak{C}$, which equated to F/C will give the rigorous equation to the required surface; but the result would be so complex as to be of little value because not easily intelligible.

I therefore at once proceed to approximation.

S being very large compared with M and m, ρ will be small compared with r.

Then since $$r^2 = R^2 + \rho^2 + 2R\rho\cos\theta,$$

$$\frac{R^3}{r^3} = 1 - 3\frac{\rho}{R}\cos\theta - \frac{3}{2}\frac{\rho^2}{R^2} + \frac{15}{2}\frac{\rho^2}{R^2}\cos^2\theta.$$

Therefore approximately

$$\mathfrak{P} = -\frac{S}{R^2}\left\{\frac{\rho}{R}\left(1 - \frac{3\rho}{R}\cos\theta\right)\right.$$
$$\left. + \left(-\frac{3\rho}{R}\cos\theta - \frac{3}{2}\frac{\rho^2}{R^2} + \frac{15}{2}\frac{\rho^2}{R^2}\cos^2\theta\right)\cos\theta\right\}$$
$$= -\frac{S\rho}{R^3}\left\{1 - 3\cos^2\theta - \frac{9}{2}\frac{\rho}{R}\cos\theta + \frac{15}{2}\frac{\rho}{R}\cos^3\theta\right\},$$

and

$$\mathfrak{P}^2 = +\frac{S^2\rho^2}{R^6}\left\{(1 - 3\cos^2\theta)^2\right.$$
$$\left. + 3\cos\theta\,(5\cos^2\theta - 3)(1 - 3\cos^2\theta)\frac{\rho}{R}\right\},$$

Again

$$\mathfrak{T} = -\frac{S\rho}{R^3}\left\{3\cos\theta + \frac{3}{2}\frac{\rho}{R} - \frac{15}{2}\frac{\rho}{R}\cos^2\theta\right\}\sin\theta,$$

and

$$\mathfrak{T}^2 = +\frac{S^2\rho^2}{R^6}\left\{9\cos^2\theta\,(1 - \cos^2\theta)\right.$$
$$\left. - 9\cos\theta\,(5\cos^2\theta - 1)\,(1 - \cos^2\theta)\frac{\rho}{R}\right\},$$

whence

$$\mathfrak{F}^2 = \frac{S^2\rho^2}{R^6}\left\{1 + 3\cos^2\theta - 12\cos^3\theta\frac{\rho}{R}\right\}.$$

Now

$$\mathfrak{C} = \frac{M+m}{\rho^2}.$$

And

$$\frac{\mathfrak{F}^2}{\mathfrak{C}^2} = \left(\frac{S}{M+m}\right)^2\frac{\rho^6}{R^6}\left\{1 + 3\cos^2\theta - 12\cos^3\theta\frac{\rho}{R}\right\}.$$

We now have to introduce a similar approximation into the value of F^2/C^2.

We have $$\frac{r^4}{R^4} = 1 + 4\frac{\rho}{R}\cos\theta,$$

and therefore $$\frac{F^2}{C^2} = \left(\frac{M}{S+m}\right)^2 \frac{R^4}{\rho^4}\left(1 + 4\frac{\rho}{R}\cos\theta\right).$$

Equating F^2/C^2 to $\mathfrak{F}^2/\mathfrak{C}^2$, we get

$$\left(\frac{R}{\rho}\right)^{10} = \left(\frac{S}{M}\right)^2\left(\frac{S+m}{M+m}\right)^2\left\{1 + 3\cos^2\theta - 4\cos\theta\,(1 + 6\cos^2\theta)\frac{\rho}{R}\right\};$$

$$\therefore\quad \frac{R}{\rho} = \left(\frac{S}{M}\right)^{\frac{1}{5}}\left(\frac{S+m}{M+m}\right)^{\frac{1}{5}}(1 + 3\cos^2\theta)^{\frac{1}{10}}\left\{1 - \frac{2}{5}\,\frac{\cos\theta\,(1 + 6\cos^2\theta)}{1 + 3\cos^2\theta}\,\frac{\rho}{R}\right\}.$$

Thus the equation to the surface is approximately

$$\frac{R}{\rho} = \left(\frac{S}{M}\right)^{\frac{1}{5}}\left(\frac{S+m}{M+m}\right)^{\frac{1}{5}}(1 + 3\cos^2\theta)^{\frac{1}{10}}\left\{1 - \frac{2}{5}\left(\frac{M}{S}\right)^{\frac{1}{5}}\left(\frac{M+m}{S+m}\right)^{\frac{1}{5}}\frac{\cos\theta\,(1 + 6\cos^2\theta)}{(1 + 3\cos^2\theta)^{\frac{11}{10}}}\right\}.$$

It is usually the case that m is negligible compared with M, and that M is also small compared with S, and in this case we may write the equation with sufficient accuracy

$$\frac{R}{\rho} = \left(\frac{S}{M}\right)^{\frac{2}{5}}(1 + 3\cos^2\theta)^{\frac{1}{10}}.$$

Laplace gives a formula for the radius of the sphere of activity which is virtually derivable from the above investigation on the special hypothesis that the three bodies lie in a straight line. Thus he puts θ equal to zero or 180° and finds,

$$\frac{R}{\rho} = 4^{\frac{1}{10}}\left(\frac{S}{M}\right)^{\frac{2}{5}}.$$

But to find the true mean value of $(1 + 3\cos^2\theta)^{\frac{1}{10}}$, we must estimate it all over the sphere.

Now

$$\frac{1}{4\pi}\iint(1 + 3\cos^2\theta)^{\frac{1}{10}}\sin\theta\,d\theta\,d\phi = \int_0^1 (1 + 3x^2)^{\frac{1}{10}}\,dx.$$

This integral evaluated by quadratures, is found to be equal to 1·063.

Thus the true mean gives

$$\frac{R}{\rho} = 1{\cdot}063\left(\frac{S}{M}\right)^{\frac{2}{5}}.$$

Laplace makes it

$$\frac{R}{\rho} = 4^{\frac{1}{10}}\left(\frac{S}{M}\right)^{\frac{2}{5}} = 1{\cdot}149\left(\frac{S}{M}\right)^{\frac{2}{5}}.$$

The ratio of the least to the greatest value of ρ in the formula suggested in this note is 1·149, and Laplace takes the minimum value of ρ as the radius of his sphere.

In the case of Jupiter, Laplace's formula gives $\rho = {\cdot}054\,R$, and my formula gives $\rho = {\cdot}058\,R$.

It follows that Laplace's conclusion is sufficiently accurate for the purpose for which it is intended.

(3) *The change of zero of Thermometers.* By C. T. HEYCOCK, M.A., King's College.

The author described the result of experiments he had made in conjunction with Mr Neville to overcome the change in zero which thermometers undergo when heated for a long time. The method consisted in boiling the thermometer for eighteen days in baths of either mercury or sulphur, at the end of this time the zeros were found to be practically fixed unless they were exposed to higher temperatures than those of the substance in which they were boiled. The paper was illustrated by a curve showing that the change in zero was very rapid for the first few hours, amounting in a special case to 11° C. for 20 hours heating, but that afterwards the change became almost nil as the heating was continued.

(4) *The Elasticity of Cubic Crystals.* By A. E. H. LOVE, M.A., St John's College.

(5) *Changes in the dimensions of Elastic Solids due to given systems of forces.* By C. CHREE, M.A., Fellow of King's College.

[*Abstract.*]

This paper deduces from a general theorem due to Professor Betti expressions for the mean values of the strains and stresses in any homogeneous elastic solid acted on by any given system of bodily and surface forces. Formulae for the mean strains in isotropic solids acted on only by surface forces were given by

Professor Betti, but he does not seem to have considered the general case, nor to have made applications such as those treated here. From the formulae for the mean strains the change can be found in the mean length, taken over the cross section, of any right cylinder or prism subjected to any given system of forces. Similarly the change in the whole volume of any elastic solid of any shape can always be expressed as the sum of a volume and a surface integral involving only the applied forces and the elastic constants of the material.

Thus in an isotropic solid acted on by bodily forces whose components are X, Y, Z per unit *volume*, and by surface forces whose components are F, G, H per unit surface, the change δv in the volume is given by

$$3k\delta v = \iiint (Xx + Yy + Zz)\, dxdydz + \iint (Fx + Gy + Hz)\, dS,$$

where k denotes the bulk modulus, or $m - \frac{1}{3}n$ in Thomson and Tait's notation. It is obvious from the equations of statical equilibrium that the position of the origin in the above expressions is immaterial. In any homogeneous aeolotropic solid the change in volume may be similarly determined, but the expressions under the integral signs are a little longer.

The several formulae both for isotropic and aeolotropic solids are applied to a variety of special cases, a few of which will serve for illustration. The material to which the following results apply is, unless otherwise stated, assumed isotropic.

When a solid of any shape is suspended from a point, or a series of points in one horizontal plane, its volume v is greater than if "gravity" did not act, and the increment δv due to "gravity", represented by g, is given by

$$\delta v/v = g\rho h/3k,$$

where ρ is the density and h the distance of the centre of gravity below the point, or points, of suspension. On the other hand, if a body be supported on a smooth plane, or at a series of points in a horizontal plane, its volume is diminished owing to the action of gravity, the diminution $(-\delta v')$ being given by

$$-\delta v'/v = g\rho h'/3k,$$

where h' is the height of the centre of gravity above the plane of support.

When a right cylinder or prism is suspended with its axis vertical its length l is increased, and the mean increment δl taken over its cross section is given by

$$\delta l/l = g\rho l/2E,$$

where E is Young's modulus. When the cylinder rests on a smooth horizontal plane with its axis vertical, it shortens under

gravity by an amount equal to the above. When the cylinder is suspended with its axis horizontal, in such a way that bending does not occur, it shortens, while when supported on a smooth horizontal plane in that position it lengthens. The alterations in the mean length in the two cases are given by

$$\delta l/l = \mp \eta g\rho h/E,$$

where η is Poisson's ratio, while h is the distance of the centre of gravity from the horizontal plane through the points of suspension in the first case and through the points of support in the second.

When a solid of any shape rotates with uniform angular velocity ω about a principal axis of inertia through its centre of gravity the volume v is increased, the increment being given by

$$\delta v = \omega^2 I/3k,$$

where I is the moment of inertia of the body about the axis of rotation.

When a right cylinder or prism rotates about its axis it shortens, and the mean shortening $(-\delta l)$ taken over the cross section is given by

$$-\delta l/l = \eta\omega^2\rho\kappa^2/E,$$

where κ is the radius of gyration of the cross section about the axis.

When a rectangular parallelepiped $2a \times 2b \times 2c$ rotates about the axis $2c$, the mean increment $2\delta a$ in the distance between the faces perpendicular to $2a$ is given by

$$\delta a/a = \omega^2\rho\,(a^2 - \eta b^2)/3E.$$

Thus the tendency to increase in length in material lines perpendicular to the axis of rotation becomes reversed when the dimension perpendicular to this and to the axis of rotation is sufficiently great.

A homogeneous sphere, whether isotropic or aeolotropic, owing to the mutual gravitation of its parts suffers a diminution in volume given by

$$-\delta v/v = g\rho R/5k,$$

where R is the radius and g "gravity" at the surface. This suffices to prove that the application of the mathematical theory of elasticity to the earth, treated as a homogeneous solid, violates the fundamental condition that the strains must be small, unless the material be assumed to offer a much greater resistance to compression than any known material under normal conditions at the earth's surface.

The change in volume due to the mutual gravitation in its parts in any very nearly spherical body, when isotropic, is shown

to be the same as in a sphere of the same material of equal volume, and it is thence concluded that the spherical is a form in which the reduction of volume due to gravitation is in general either a maximum or a minimum. The reduction of volume is calculated for a gravitating ellipsoid, and it appears that the sphere is the form in which, when the volume is given, the reduction is a maximum. In a nearly spherical ellipsoid whose principal sections through the longest axis are of eccentricities ϵ_1 and ϵ_2 the reduction in volume is given by

$$-\delta v/v = (g\rho R/5k)\{1 - (\epsilon_1^4 - \epsilon_1^2\epsilon_2^2 + \epsilon_2^4)/45\},$$

where R is the radius, and g the value of gravity at the surface, in a sphere of equal volume and density.

In a given volume of an aeolotropic material a very slight assumption of an ellipsoidal form, insufficient to produce an appreciable effect if the material were isotropic, increases or diminishes the diminution in volume due to mutual gravitation according as it consists in a lengthening or a shortening of those material lines which are parallel to directions in which the linear contraction under uniform normal pressure is above the average.

(6) *On the law of distribution of velocities in a system of moving molecules.* By A. H. LEAHY, M.A., Pembroke College.

1. The following proof appears briefly to establish the fact that Maxwell's law of distribution of velocities gives the only steady distribution. The proof is a little shorter than the ordinary proof as given by Boltzmann, even if Mr Burbury's variation of it as published in the *Philosophical Magazine* for October 1890 be adopted.

Let a particle whose velocity is OP in magnitude and direction strike a particle whose velocity is op. Suppose the particles to belong to different systems, and let the number of particles of the first kind which have velocity components lying between ξ and $\xi + d\xi$, η and $\eta + d\eta$, ζ and $\zeta + d\zeta$, where ξ, η, ζ are the components of OP, be $F(OP)\,d\xi d\eta d\zeta$. Let $f(op)\,d\xi' d\eta' d\zeta'$ have a similar meaning when applied to the particles of the second system. Then the number of impacts which particles with velocity OP have with particles of the second system which have velocity op will, in the interval dt, be per unit volume

$$F(OP)\,d\xi d\eta d\zeta \,.\, \pi s^2 u dt \,.\, f(op)\,d\xi' d\eta' d\zeta' \quad \text{.........(1)},$$

since each particle in the unit volume strikes, on the average, $\pi s^2 u dt f(op)\,d\xi' d\eta' d\zeta'$ in the interval dt; the particles being regarded as hard spheres, the sum of the radii of two spheres, one

of each system, being s, and u the velocity of a particle of the first system relative to the velocity of a particle of the second system.

Suppose now that after encounter the velocities of the particles become OP_1, op_1 respectively, so that the components of OP_1 lie between ξ_1 and $\xi_1 + d\xi_1$, η_1 and $\eta_1 + d\eta_1$, ζ_1 and $\zeta_1 + d\zeta_1$; and ξ_1', η_1', ζ_1' have similar meanings as the components of op_1. Since the velocity of the centre of gravity is unchanged by the impact, the condition that the required change shall take place is that the direction of u shall lie within a cone of solid angle dS making an angle θ with the line of centres at impact. A collision such that velocities OP, op before collision may become velocities OP_1, op_1 after collision may be called a "collision of a given kind," and since, as Mr Burbury has pointed out, all directions of the relative velocity after encounter are equally probable if the molecules behave as hard elastic spheres, the whole number of encounters of the given kind in the interval dt will be

$$F(OP)f(op)\,d\xi d\eta d\zeta\,.\,d\xi' d\eta' d\zeta'\,.\,\pi s^2 u\,dt\,.\,\frac{dS}{4\pi}\,\ldots\ldots(2)$$

per unit of volume.

Conceive now that the velocity of every molecule of the system is suddenly reversed in sign, the molecules of the solid boundary of the system, if such exist, being similarly reversed as to the direction of their velocities, the position of every molecule being unaltered. The physical properties of such a medium will of course differ in many respects from those of the original medium, but it will at any rate have this property, that all particles whose velocities in the original medium changed from OP to OP_1 in the time dt will in the second (or as we may call it the "reversed") medium change in a second interval dt from $-OP_1$ to $-OP$. Now, since the distribution of particles is perfectly regular in space, the distribution of velocities in the original medium "behind" any class of molecules is the same as the distribution "in front" of the same class. Hence the number which in the reversed medium change their velocity $-OP_1$ for $-OP$ by striking particles with velocity op_1 is in the interval dt

$$F(-OP_1)f(-op_1)\,d\xi_1 d\eta_1 d\zeta_1\,.\,d\xi_1' d\eta_1' d\zeta_1'\,.\,\pi s^2 u' dt\,.\,\frac{dS'}{4\pi}\ldots(3),$$

where u' is the velocity of OP_1 measured relatively to op_1 and dS' is the angle of the cone, whose axis makes an angle θ with the line of centres, within which the direction of the relative velocity u' must lie in order that the collision may be one of the given kind. Since the number which change from velocities OP to velocities OP_1 in the original medium by collisions of the given

kind is equal to the number which change from OP_1 to $-OP$ in the reversed medium by collisions of the same kind, expressions (2) and (3) are equal. Also u' in (3) is the same as u in (2) since the velocity of the centre of mass is unchanged and the spheres are elastic, and by ordinary geometry dS' is equal to dS and $d\xi d\eta d\zeta . d\xi' d\eta' d\zeta'$ is equal to $d\xi_1 d\eta_1 d\zeta_1 . d\xi_1' d\eta_1' d\zeta_1'$. Hence, since $F(-OP_1)$ in the reversed medium is equal to $F(OP_1)$ in the original one,

$$F(OP_1) f(op_1) = F(OP) f(op),$$

and therefore since the kinetic energy is unchanged by the impact we have, by the ordinary methods, Maxwell's law of distribution namely

$$F(OP) = Ae^{-\frac{OP^2}{\alpha^2}}.$$

In order to examine the validity of the above proof the assumptions underlying equations (1) and (2) should be further considered. In equation (1) the assumption is that, if a number of particles are distributed uniformly throughout a medium, and if the velocities are so distributed that the number per unit volume which have velocity op is $f(op)\, d\xi' d\eta' d\zeta'$, then the number in the volume $F(OP)\, d\xi d\eta d\zeta . \pi s^2 u dt$, which may be written do_1, is $f(op)\, d\xi' d\eta' d\zeta' . do_1$. This assumption is equivalent to two others, first, that the particles are distributed throughout the medium with perfect uniformity so that we can safely take the number of molecules of a given kind in an element of volume to be proportional to the volume of the element if the element is large enough to contain a great many molecules; secondly, that the particular volume do_1 is large enough for the first assumption to be applied to it. In order to prove result (2) we must suppose that the number of molecules within the volume

$$F(OP)\, d\xi d\eta d\zeta . \pi s^2 u dt . \frac{dS}{4\pi} \cos\theta,$$

which may be written do_2, is proportional to do_2. Thus, since the volume do_1 is greater than do_2, the whole assumption that we make is that do_2, and consequently do_1, is large enough to contain a very large number of the molecules which are distributed so that the number which have a velocity op is given by the function $f(op)$.

To estimate the number of these molecules, suppose $F(OP)$ to be equal to $\frac{N}{\pi^{\frac{3}{2}}\alpha^3} e^{-\frac{OP^2}{\alpha^2}}$, which is Maxwell's law of distribution. Then, taking hydrogen as an example, since* $N\pi s^2$ is $7{\cdot}0 \times 10^3$

* These numbers are calculated in accordance with Professor Tait's results, *Edin. Trans.* XXXIII. p. 91.

approximately, the volume is

$$\frac{7\cdot 0 \times 10^3}{\pi^{\frac{3}{2}}} e^{-\frac{OP^2}{\alpha^2}} \cdot \frac{d\xi d\eta d\zeta}{\alpha^3} \cdot udt \cdot \frac{dS}{4\pi} \cos\theta$$

cubic millimetres. Now a cubic millimetre of hydrogen at atmospheric pressure contains about $9\cdot 76 \times 10^{16}$ molecules. Hence the number of molecules in do_2 is

$$6\cdot 8 \times 10^{20} \cdot e^{-\frac{OP^2}{\alpha^2}} \cdot \frac{d\xi d\eta d\zeta}{\alpha^3} \cdot udt \cdot \frac{dS}{4\pi} \cos\theta.$$

Suppose u to be equal to k times α which is in hydrogen equal to $7\cdot 62 \times 10^5$ millimetres per second; we get the whole number of molecules in do_2 to be

$$5\cdot 2 \cdot e^{-\frac{OP^2}{\alpha^2}} \cdot \frac{d\xi d\eta d\zeta}{\alpha^3} \cdot 10^{26} kdt \frac{dS}{4\pi} \cdot \cos\theta,$$

dt being measured in seconds; and this number must be large in order that equation (2) may accurately give the number of collisions of the given kind.

The smallest value which we can ascribe to dt will depend upon the magnitude of the limits $d\xi$, $d\eta$, $d\zeta$, dS which define the encounter. Suppose that $d\xi = d\eta = d\zeta = \alpha/1000$; suppose also that $dS/4\pi = 10^{-3}$. Let us also suppose OP not to be greater than 2α, a supposition which excludes less than $0\cdot 5$ per cent. of the whole number of molecules. Suppose also that k is greater than 10^{-2}, so that the relative velocity u is not less than $\alpha/100$. These suppositions give the whole number of molecules in do_2 to be greater than $9\cdot 6\, dt \cdot 10^{10}$, so that this number is more than a million if dt is not less than 10^{-5} of a second. This estimate of the limiting value of dt is perhaps too small as we have taken the limits of $d\xi d\eta d\zeta$ exceedingly small, but it will appear that dt must not be taken indefinitely small and should at any rate be greater than the mean time between collisions, which is of the order 10^{-9} of a second. With the above proviso as to the value of dt, it appears that result (2) can be taken to be accurate, and since result (3) is merely an application of result (2) to the reversed medium it appears that the assumptions made can be relied on.

2. It has throughout been assumed that the distribution of velocities is "steady", and the proof shows that Maxwell's law gives the only possible steady distribution. It is however desirable to show if possible that the system must ultimately acquire a steady distribution. Now the steadiness of the distribution has been assumed twice, first when the assertion is made that the

number of molecules with velocity op struck by molecules moving among them with relative velocity u is proportional to $u\,dt\,.\,\pi s^2$, secondly, when the distribution in the "reversed" medium was taken to be the same as that in the original one. If the distribution is not steady, expression (2) must be amended by inserting a factor $(1+v\,dt)$, and expression (2) must contain a factor $(1+v'dt)$; where v, v' depend upon the variations of F and f. The result obtained as before will be that

$$F(OP_1)f(op_1)-F(OP)f(op)$$

will not be zero but equal to $w\,dt$ where w depends upon v, v', F, and f. Integrating equation (2) and using the proposition that all directions of encounter are equally probable, we get the usual result

$$\frac{d}{dt}F(OP)=F(OP)\,d\xi d\eta d\zeta\iiint d\xi' d\eta' d\zeta'\,\pi s^2 u$$
$$\frac{1}{4\pi}\left\{\iint\Big(F(OP_1)\,f(op_1)-F(OP)\,f(op)\Big)\,dS\right\},$$

where $d\xi'$, $d\eta'$, $d\zeta'$ are elementary increments of the components of op, and the double integral is taken for all possible impacts between particles whose velocities before collision were OP, op; OP_1, op_1 being the velocities which the particles acquire if their relative velocity falls within the cone of solid angle dS.

Since the subject of integration in the double integral is equal to $w\,dt$, $\frac{d}{dt}F(OP)$ must contain dt as a factor and will be very small when dt is very small. But, since there is a limit to the minimum value of dt, this does not prove $\frac{d}{dt}F(OP)$ to be zero; that is we cannot in this way prove the ultimate distribution to be steady, although its variation from the steady state must be small when the distribution of the particles is regular throughout the space considered.

Boltzmann's proof would show that the function H which he has introduced will continually diminish until the steady state is obtained, but I think that it assumes equations (1) and (2) to be absolutely true, which they appear to be if the motion is from the first assumed to be steady. The proof that the motion of the particles finally must attain a steady state is apparently still wanting, although the above argument shows that the divergence from the steady state must ultimately be small. It is not impossible that $F(OP)$ may ultimately be periodic with a period of magnitude of the same order of magnitude as the time of free path. But the assumption that the motion of the particles is

ultimately absolutely steady is after all not greater than the assumption that it is ultimately perfectly regular, and if the regularity of the distribution both in space and time is assumed Maxwell's law of distribution appears, from the above, readily to follow.

Monday, *May* 2, 1892.

Prof. G. H. Darwin, President, in the Chair.

The following were elected Fellows of the Society:

Thomas Clifford Allbutt, M.D., F.R.S., Fellow of Caius College, Regius Professor of Physic.

David Sharp, M.A. (M.B. Edin.), F.R.S., Curator in Zoology.

J. C. Willis, B.A., Gonville and Caius College.

The following was elected an Associate:

A. Antunis Kanthack (M.B. Lond.), St John's College, John Lucas Walker Student in Pathology.

The following communications were made to the Society:

(1) *The application of the Spherometer to Surfaces which are not Spherical.* By J. Larmor, M.A., St John's College.

The ordinary form of spherometer, which is used for measuring the curvatures of lenses, rests on the surface to be measured by three legs which are at the corners of an equilateral triangle; and the mode of using it consists in finding the length of the ordinate drawn up to the surface from the centre of the triangle formed by the points of support, by means of a micrometer screw moving along the axis of the instrument.

In the actual use of the instrument the surface to be measured is assumed to be spherical; and the question has apparently not occurred to examine the character of the results which may be derived from its application to a surface of double curvature.

On actual trial with such a surface, for example the cylindrical surface of an iron pipe, it appears at once that when the centre is set at a given point the instrument may be rotated anyhow on its axis without affecting its reading. It therefore measures some definite quality of the double curvature of the surface at the point. There is a temptation to hastily assume that the plane of support is parallel to the tangent plane at the centre of the instru-

ment, that it is in fact the indicatrix plane of that point, and to deduce that the reading gives the mean of the principal curvatures of the surface; this result is correct, but the assumption just mentioned is erroneous.

To obtain a rigorous investigation, let us assume that the points of support form an isosceles triangle, let the base subtend an angle 2α at the centre of the circumscribing circle, and let c be the radius of this circle and h the ordinate drawn from its centre up to the surface. If this ordinate is taken as axis of z, the equation of the surface will be

$$z = px + qy + \frac{x^2}{2R_1} + \frac{y^2}{2R_2},$$

where $(p, q, -1)$ is the direction of the tangent plane at the origin, and R_1, R_2 are the radii of principal curvature. As the three legs rest on the surface, we have

$$h = cp\cos(\theta+\alpha) + cq\sin(\theta+\alpha) + \tfrac{1}{2}c^2\left\{\frac{\cos^2(\theta+\alpha)}{R_1} + \frac{\sin^2(\theta+\alpha)}{R_2}\right\},$$

$$h = cp\cos(\theta-\alpha) + cq\sin(\theta-\alpha) + \tfrac{1}{2}c^2\left\{\frac{\cos^2(\theta-\alpha)}{R_1} + \frac{\sin^2(\theta-\alpha)}{R_2}\right\},$$

$$h = -cp\cos\theta \qquad - cq\sin\theta \qquad + \tfrac{1}{2}c^2\left\{\frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2}\right\},$$

where $\pi + \theta$ is the azimuth of the vertex of the triangle of support. We are to eliminate p, q, and so connect h with R_1, R_2 and θ. By addition of the first pair of relations

$$2h = 2cp\cos\theta\cos\alpha + 2cq\sin\theta\cos\alpha + \tfrac{1}{2}c^2\left\{\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\cos 2\theta\cos 2\alpha\right\};$$

therefore by use of the third

$$2h(1+\cos\alpha) = \tfrac{1}{2}c^2\left\{\left(\frac{1}{R_1} + \frac{1}{R_2}\right)(1+\cos\alpha) + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\cos 2\theta(\cos 2\alpha + \cos\alpha)\right\},$$

or on rejecting the factor $1 + \cos\alpha$,

$$\frac{4h}{c^2} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\cos 2\theta(2\cos\alpha - 1).$$

The value of h therefore depends on the azimuth θ except in one case, when α is $\tfrac{1}{3}\pi$ so that the triangle of support is equilateral, which is the case referred to above. The quantity involved

in the formula is then $\frac{1}{R_1}+\frac{1}{R_2}$; and by referring back to the original case of a spherical surface we see that the instrument measures the arithmetic mean of the principal curvatures.

Thus for example the equilateral form of the instrument may be conveniently used to measure the curvature of a cylindrical lens or a cylindrical pipe, but for that purpose its indication must be doubled.

The equilateral form will be of no use for testing deviation from sphericity at a given point of a surface. The isosceles form may however be so used, the difference of the extreme curvature-indications given by it for any point being by the above formula

$$\left(\frac{1}{R_1}-\frac{1}{R_2}\right)(2\cos\alpha-1),$$

that is directly proportional to the difference of the principal curvatures. The curvature may thus be completely explored*.

In all these formulae the usual assumption is made that the span of the instrument is small compared with the radii of curvature of the surface.

If the instrument had four legs at the corners of a rectangle, there would be only two positions in azimuth, corresponding to the sections of greatest and least curvature, in which it would rest firmly at a given point on a surface, with all its legs in contact; and the plane of contact would in this case be parallel to the tangent plane at the summit of the surface. The readings for these positions would give

$$\frac{\cos^2\alpha}{R_1}+\frac{\sin^2\alpha}{R_2}$$

and

$$\frac{\sin^2\alpha}{R_1}+\frac{\cos^2\alpha}{R_2},$$

where α is an angle made by a diagonal of the rectangle with a side; so that the values of both principal curvatures might thus be determined.

* Mr H. F. Newall informs me that an isosceles spherometer is used by Dr Common for exploring the curvatures of his large specula.

May 16, 1892.

PROFESSOR G. H. DARWIN, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

W. Robertson Smith, M.A., Fellow of Christ's College, Professor of Arabic.
J. K. Murphy, B.A., Caius College.

The following communications were made to the Society:

(1) *Recent advances in Astronomy with Photographic Illustrations.* *By* H. F. NEWALL, M.A., Trinity College.

A series of photographs was exhibited by the lantern and described, to illustrate recent progress in astronomical photography. The series included some interesting specimens taken with the Newall telescope, in which the object glass is not specially corrected for photographic purposes.

(2) *On the pressure at which the electric strength of a gas is a minimum.* *By* J. J. THOMSON, M.A., Cavendish Professor.

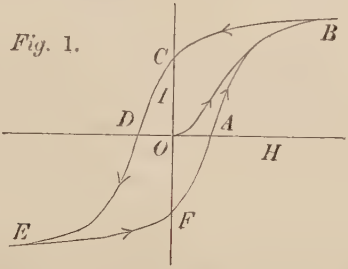
The author showed that when no electrodes are present, the discharge passes through air at a pressure somewhat less than that due to 1/250 mm. of mercury; the discharge passes with greater ease than it does at either a higher or a lower pressure. Mr Peace has lately shown that when electrodes are used, the critical pressure may be as high as that due to 250 mm. of mercury: so that as the spark length is altered the critical pressure may range from 250 mm. to 1/250 of a mm. It was pointed out that this involved the possession by a gas conveying the discharge of a structure much coarser than any recognized by the Kinetic Theory of Gases. The author suggested a theory of such a structure and showed that the theory would account for the influence of spark length and pressure on the potential difference required to produce discharge.

(3) *On a compound magnetometer for testing the magnetic properties of iron and steel.* By G. F. C. SEARLE, M.A., Peterhouse.

When a bar of iron or steel is subjected to the action of a longitudinal magnetic force, H, it is found that the intensity of magnetisation of the iron thereby produced depends not only upon the value of H at the instant, but also upon the series of

values which H has previously assumed. Thus if the magnetising force is made to undergo a series of changes in a cyclical manner, the curve representing the relation of I to H will be of the form of a loop, which however degenerates into a straight line when the maximum value of H does not exceed ·04 C.G.S. units*. This dependence of the intensity of magnetisation, due to a given magnetic force, upon the previous magnetic history of the iron has been called by Ewing *hysteresis*. The curve $ABCDEFA$ (fig. 1)

Fig. 1.



Hysteresis Curve for Annealed Steel Wire.

taken from Prof. Ewing's book on "Magnetic Induction in Iron and Other Metals" will serve to give a general idea of the relation between I and H for a piece of annealed pianoforte steel wire when the magnetic force is made to pass repeatedly through a complete cycle of changes. The maximum values of H and I in this curve are about 100 and 1100 C.G.S. units respectively. The curve OB gives the relation between I and H for a piece of steel which has never previously been magnetised or which has been completely demagnetised by continued reversals of a magnetic force whose amplitude has been slowly diminished to zero.

These hysteresis curves are of great interest from the practical as well as from the philosophical point of view, since, as has been shown by Warburg and by Ewing†, the area of the curve, when estimated on the proper scale, represents the energy expended per cubic centimetre of the iron in carrying H through its cycle of changes.

In determining the form of the hysteresis curve, a specimen of the material in the form of a wire is placed inside a long uniformly wound solenoid through which a current can be sent. The current gives rise, inside the solenoid, to a uniform longitudinal magnetic force whose value can be calculated from the equation

$$H = 4\pi ni,$$

* Lord Rayleigh, *Phil. Mag.* March, 1887.

† J. A. Ewing, "Magnetic Induction in Iron and Other Metals," § 79.

where i is the strength of the current, and n the number of turns of wire upon the solenoid per centimetre of its length. The solenoid is placed near a suitable mirror magnetometer; and a small coil, which is joined up in series with the magnetising solenoid, is so adjusted that it exactly neutralises the action of the *solenoid* itself upon the magnetometer. Thus when the specimen of iron is placed inside the solenoid the deflection produced is due entirely to the magnetisation of the specimen. From the observed deflection the value of the intensity of magnetisation, I, can be determined. The magnetising current also passes round a suitable galvanometer by means of which its strength, i, can be measured. The strength of the current is gradually varied by means of a resistance box in the circuit, and the simultaneous readings of the galvanometer and magnetometer are noted. The values of H and I deduced from these readings are used as abscissa and ordinate in the construction of the hysteresis curve. This process naturally involves a good deal of labour.

I have endeavoured to construct an instrument which should perform simultaneously the functions of both galvanometer and magnetometer and should cause a spot of light to trace out a hysteresis curve upon a screen. One method of attaining this end is to provide a mirror with two independent motions about two axes mutually at right angles, the motions about these two axes being governed by two small magnets. One of these magnets must be acted on by a magnetic force proportional to the magnetising current, and the other by a magnetic force proportional to the intensity of magnetisation of the specimen.

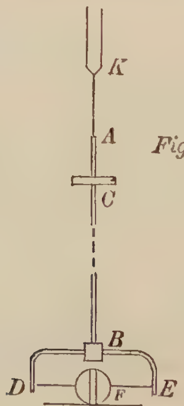


Fig. 2.

This idea was put into practice in the following manner. AB (fig. 2) is a thin aluminium wire about 80 centimetres long. This is suspended by one end A by a silk fibre from the support K.

Near its top the wire carries a small magnet C whose axis is at right angles to the wire. The lower end B of the wire carries a small fork DBE, also of aluminium wire, across which the silk fibre DE is stretched. Attached to this fibre by means of wax is a small plane mirror F, such as is used in reflecting galvanometers, carrying a small magnet whose axis is at right angles to the fibre DE. Attached to the bottom edge of the mirror is a disk of thin mica about 1 inch in diameter. When the plane of the mirror is vertical, the plane of the disk is horizontal. Close beneath the mica disk is placed a piece of cardboard in a horizontal position. The mica disk, owing to the close proximity of the cardboard, very rapidly reduces the mirror to rest. The mirror is fixed to the fibre so that the centre of gravity of the mirror and mica disk is slightly below the line DE. Thus the controlling force acting on the lower system consists partly of gravity and partly of the magnetic force due to the earth and to any control magnets which may be required to bring the mirror into any desired position. The mirror now possesses two independent motions, the one about the axis AK and the other about the axis DE.

The apparatus is set up as in fig. 3*, in which the suspended part has been turned through a right angle so that the mirror is

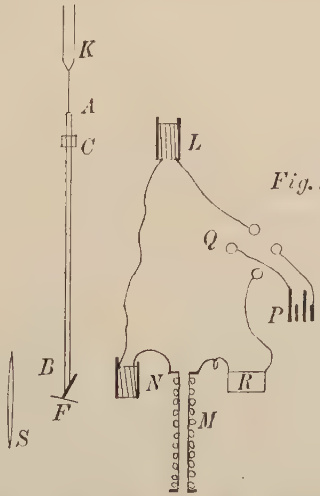


Fig. 3.

now seen edgeways. The plane of the paper is supposed to be a plane through the wire AB at right angles to the magnetic meridian. The magnet C is therefore at right angles to the plane of the paper. The mirror F is shown slightly tilted. The coil L

* This figure is purely diagramatical and does not represent the relative proportions of the separate parts of the apparatus.

is placed near the magnet C and deflects it through an angle proportional to the strength of the current in L, the deflections being kept very small. The solenoid M is placed in a vertical position east or west of the mirror, its upper end being about in a horizontal line with the mirror. The small coil N can be adjusted so that the effect on the magnet F of the solenoid itself is completely neutralised. R is a resistance box for varying the current, P a battery, and Q a commutator. A lens S of about 40 inches focal length forms the window of the case in which the suspended part is hung. A lamp and screen are placed at about 40 inches from the lens. Cross wires are placed in front of the lamp and a sharp image of these is thrown upon the screen by the action of the lens and mirror. The spot of light may be made to take up any desired position on the screen by properly adjusting small permanent magnets in the neighbourhood of the two magnets C and F. I had expected that a good deal of trouble would have been caused by change of zero in the vertical direction owing to changes in the silk fibre on which the mirror is strung, but I was agreeably surprised to find that the spot of light would, if the mirror were disturbed, return to the same horizontal position to within $\frac{1}{50}$ inch. The zero position seemed to be quite permanent.

In order that the two motions of the spot of light should take place in horizontal and vertical lines, the axis DE must be adjusted so as to be accurately perpendicular to the axis of suspension, AK. The necessary fine adjustment is easily made by slightly bending the suspending wire near the point B. I found that a small block of cork formed the best means of connecting the wire AB with the fork DBE. To get rid of any secondary effect of the coil L upon the lower magnet a second small "compensating" coil may be included in the circuit. In order to bring the spot of light quickly to rest a suitable mica vane was attached to the vertical wire AB. This rapidly stops the motions in azimuth.

When I exhibited the instrument to the Society, the magnet C was slightly affected by the induced magnetization of the specimen of iron in the solenoid. This effect can not be compensated by another coil, since a coil through which the magnetising current flows will not imitate the magnetic behaviour of the iron. To remedy this defect I have fitted a second magnet of moment nearly equal to that of C to the vertical wire a short distance below C, its axis pointing in the opposite direction to that of C. The effect of the magnetised specimen on the astatic system is very small and I hope that all trouble from this source has now been got rid of.

The indications of the instrument can easily be reduced to absolute measure (at least approximately) in the following way.

Suppose, for instance, that it is desired that a movement of the spot of light through 10 centimetres horizontally should correspond to a magnetic force inside the solenoid equal to 100 C.G.S. units. A known current is sent round the coil L, and this coil is then so adjusted that the spot of light is deflected through the proper distance corresponding to the calculated value of H. To standardise the vertical motion of the spot, a magnetised steel wire may be placed inside the solenoid M, through which no current is passing, and the solenoid is then adjusted until the spot of light shows a deflection in the vertical direction of 10 centimetres for each 1000 C.G.S. units of intensity of magnetisation of the steel wire. The intensity of magnetisation of the steel wire can be determined by the use of an ordinary magnetometer. If the cross section of the wire to be tested is different from that of the steel wire an appropriate factor must be introduced.

Although for very accurate observations in the subject of hysteresis the use of two separate instruments, galvanometer and magnetometer, will probably still be necessary, yet I think that the instrument I have described may be found useful as a means of rapidly gaining an approximate knowledge of the form of the hysteresis curves for various samples of iron and steel without any calculation. For this purpose it may be a useful instrument in the Lecture Room.

May 30, 1892.

PROFESSOR G. H. DARWIN, PRESIDENT, IN THE CHAIR.

The following communications were made:

(1) *The hypothesis of a liquid condition of the Earth's interior considered in connexion with Professor Darwin's theory of the genesis of the Moon.* By Rev. O. FISHER, M.A., F.G.S., Hon. Fellow of Jesus College.

IN a series of papers in the *Philosophical Transactions*, Parts I. and II. 1879, Professor Darwin has developed the theory of tidal action in the solar system.

At p. 23 of his paper on *Bodily Tides of Spheroids* he gives "a dynamical investigation of the effects of a tidal yielding of the earth on a tide of short period according to the canal theory." A numerical estimate will afford a clearer idea of the effects produced.

The symbols used are

$h =$ the depth of the canal.

$a =$ the earth's radius.

$\omega =$ the rotational speed referred to the moon (23 h. 56 m. = lunar day at present).

$\phi - \omega t =$ the longitude west of the moon.

$\epsilon =$ half the lag of the bodily tide.

$2E =$ the greatest range of the bodily tide at the equator*.

$\tau = \frac{3}{2} \times$ the moon's mass $\times a^2 \div$ (her distance)^3.

The result obtained for the height of the wave relatively to the bottom of the canal is

$$h - h\frac{d\xi}{dx},$$

where

$$\frac{d\xi}{dx} = \frac{1}{a^2\omega^2 - gh}\left\{\left(\frac{\tau}{2}\cos\epsilon - \frac{2}{5}gE\right)\cos 2(\phi - \omega t) + \frac{\tau}{2}\sin\epsilon\sin 2(\phi - \omega t)\right\}\dagger.$$

Taking up the investigation from this point since the land partakes in the rise and fall of the bottom of the canal, the measurable height of the tide will be the difference between the depth of the canal and the height of the water above the bottom of it, which will be $-h\frac{d\xi}{dx}$, or

$$\frac{-h}{a^2\omega^2 - gh}\left\{\left(\frac{\tau}{2}\cos\epsilon - \frac{2}{5}gE\right)\cos 2(\phi - \omega t) + \frac{\tau}{2}\sin\epsilon\sin 2(\phi - \omega t)\right\},$$

$$= \frac{-h}{a^2\omega^2 - gh}\frac{\tau}{2}\left\{\cos\{2(\phi - \omega t) - \epsilon\} - \frac{4}{5}\frac{gE}{\tau}\cos 2(\phi - \omega t)\right\}.$$

For $\frac{h}{a^2\omega^2 - gh}$ write H, for $2(\phi - \omega t)$ write θ.

Then the height of the tide will be expressed by

$$-H\left\{\cos(\theta - \epsilon) - \frac{4}{5}\frac{gE}{\tau}\cos\theta\right\}$$

$$= -H\left\{\cos\theta\left(\cos\epsilon - \frac{4}{5}\frac{gE}{\tau}\right) + \sin\theta\sin\epsilon\right\}.$$

* Elsewhere Prof. Darwin uses E as the ratio of the bodily tide in the case of viscosity to the like in the case of fluidity.

† Loc. cit. p. 26.

We must now estimate the numerical value of $\frac{4}{5}\frac{gE}{\tau}$.

From the definition of τ already given we have

$$\frac{4}{5}\frac{gE}{\tau}=\frac{4}{5}\frac{2}{3}\frac{(\text{distance})^3}{\text{moon's mass}}\frac{1}{a^2}\frac{\text{earth's mass}}{a^2}E,$$

$$=\frac{8}{15}\frac{\text{earth}}{\text{moon}}\left(\frac{d}{a}\right)^3\frac{E}{a},$$

$$=\frac{8}{15}\frac{1}{0\cdot01228}(60\cdot2634)^3\frac{E}{20902404}\,^*,$$

$$=0\cdot45474\,E;$$

a foot being the unit.

If the interior is considered liquid, the bodily tide may be taken equal to the equilibrium tide, which would be about $1\frac{3}{4}$ feet from highest to lowest†, and E would be half that, so that

$$\frac{4}{5}\frac{gE}{\tau}=0\cdot39789.$$

The lag of such a tide would be small. Darwin seems to consider 14′ as an admissible value for 2ϵ in that case, and $\cos\epsilon$ would be 0·9999668. Hence $\cos\epsilon$ would be greater than $\frac{4}{5}\frac{gE}{\tau}$, and we may assume

$$\frac{\sin\epsilon}{\cos\epsilon-\frac{4}{5}\frac{gE}{\tau}}=\tan D,$$

and by substitution and reduction we obtain for the height of the measurable tide

$$-H\left\{1-2\cos\epsilon\times\frac{4}{5}\frac{gE}{\tau}+\left(\frac{4}{5}\frac{gE}{\tau}\right)^2\right\}^{\frac{1}{2}}\cos\{2(\phi-\omega t)-D\},$$

or, $\cos\epsilon$ being very nearly unity,

$$-H\left\{1-\frac{4}{5}\frac{gE}{\tau}\right\}\cos\{2(\phi-\omega t)-D\},$$

$$=-H\times0\cdot60211\cos\{2(\phi-\omega t)-D\}.$$

Hence the tide would be $\frac{3}{5}$ths of what it would be on a rigid earth.

It is evident that $\tan D$ is small. Hence low water will occur a little west of the moon.

* "*The Moon*" by Proctor, Tab. IV. p. 313.
† Thomson and Tait, § 804, 2nd Ed.

In forming the potential of the protuberance of the bodily tide the earth has been taken as homogeneous. But the superficial parts having only half the mean density, it seems that the value of $\frac{4}{5}\frac{gE}{\tau}$ ought to be taken at one half that assumed, and then

$$\frac{4}{5}\frac{gE}{\tau} = 0{\cdot}19894,$$

and we find for the tide

$$-H \times 0{\cdot}80106 \cos\{2(\phi - \omega t) - D\};$$

which shows that it will be diminished by only $\frac{1}{5}$th of what its height would be if the earth was rigid.

We learn from this expression that high ocean tide will occur when $2(\phi - \omega t) - D = \pi$, that is when $\phi - \omega t = \frac{\pi}{2} + \frac{D}{2}$ to the west of the moon; and high earth tide will occur when $\phi - \omega t = -\frac{\epsilon}{2}$, or $\frac{\epsilon}{2}$ to the east of the moon. Hence the crests of the ocean and earth tides are separated by the obtuse angle $\frac{\pi}{2} + \frac{D+\epsilon}{2}$, so that the tidal protuberances of both of them, which are nearest to the moon, are to the east of it; and the effects of the couples caused by the moon's attraction upon both of them will be to retard the earth's rotation.

Prof. Darwin remarks, that the expression for the height of the ocean tide as affected by the bodily tide is subject to a modification of the same form on the equilibrium theory as on the canal theory, with the exception of a change of sign. Hence on that theory also, which neglects the inertia of the water, and therefore less nearly represents the case of nature, the ocean tide would be diminished by the same factor, and therefore only to the small extent of about one-fifth, as has been now shown would be the case on the canal theory.

The lag of the bodily tide has here been put at 14′, because Darwin has shown* that, on the hypothesis of approximate liquidity, the reaction of the moon on the bodily protuberance with that amount of lag would account for the unexplained acceleration of the moon's mean motion at the present time of 4 seconds in a century. It is evident that if the lag of the bodily tide is larger, $\cos 2\epsilon$ will be smaller, and the reduction of the tide in the canal will be still less.

* "Precession of a viscous spheroid," § 14.

The above appears to be a sufficient answer to the objection brought against the theory of internal liquidity that in such a case there could be no measurable ocean tides.

Prof. Darwin appears when he wrote to have held the view that the earth must be very rigid probably in consequence of his investigation by which he had proved that on a solid globe nothing short of a high degree of rigidity could sustain the weight of continents and mountains. This necessity is of course entirely removed by Airy's hypothesis that the crust is supported in a state of approximate hydrostatic equilibrium on a yielding nucleus*.

Assuming therefore the necessity of a high degree of rigidity, Darwin finds a certain coefficient of viscosity, which according to his calculations would cause the obliquity of the ecliptic to increase most rapidly at the present time (p. 526), and uses this particular value in his numerical calculations. Thus, when he estimates the length of time, since the moon may have been detached from the earth, at about 57 million years†, the estimate depends upon that particular value of the viscosity. So also do his estimates of the amounts of heat generated by tidal friction within the earth during certain intervals of time dating from the same epoch. And in short all the numerical results in Table IV. at p. 494, depend upon the particular assumed high degree of viscosity. It cannot therefore be too carefully borne in mind by Geologists that none of those numerical estimates, which relate to time, are applicable to the case of a liquid interior.

With respect to the obliquity of the ecliptic, it seems probable that it may have originated when the moon broke away from the earth, however much the amount of it may have since changed; for the rupture must have occurred at what was then the equator; but the alteration in the principal axes of the earth owing to its removal must have caused the axis of rotation to shift its place within the mass, so that the plane of the moon's orbit would represent that of the original equator, while the plane of the new equator would have become oblique to it.

Although, as just mentioned, the amounts of heat generated in the earth during certain intervals of time depend upon a particular assumed value for the viscosity, not so the whole amount since the rupture. Darwin says "According to the present hypothesis [of the generation of the moon] looking forward in time [from that epoch], the moon-earth system is from a dynamical point of view continually losing energy from the

* *Phil. Trans. Roy. Soc.*, vol. 145, p. 101. See also a lecture by Sir G. B. Airy "On the interior of the Earth." *Nature*, vol. 18, p. 41, 1878.

† See "Precession of a viscous spheroid and remote history of the earth." *Phil. Trans.* Pt. II. p. 531, 1879.

internal friction. One part of this energy turns into potential energy of the moon's position relatively to the earth, and the rest develops heat in the interior of the earth." It is evident therefore that, knowing the initial and present circumstances, it is possible to estimate the total amount of energy converted into heat without knowing the lapse of time in which it has occurred. Darwin finds the common period of rotation, when the moon separated from the earth, to have been 5 h. 36 m., taking the viscosity at that time as small, the earth being supposed to have been "a cooling body gradually freezing as it cools." The present rate of rotation relative to the moon (the lunar day) is 23 h. 56 m. The total heat generated in the earth in the course of this lengthening of the day if applied all at once would he says* be sufficient to heat the whole mass of the earth about 3000° Fah. supposing it to have the specific heat of iron. In Table IV. of the former paper† he had given 1760° Fah. as the temperature corresponding to a period of rotation of 6 h. 45 m., so that it appears that the additional 1240° must be due to the loss on the difference between 6 h. 45 m. and 5 h. 36 m., or 1 h. 9 m., and he remarks that, "The whole heat generated from first to last gives a supply of heat at the present rate of loss for 3560 million years. This amount of heat is certainly prodigious, and" he adds, "I found it hard to believe that it should not largely affect the underground temperature"‡; but a further calculation led him to believe that it need not do so, for he found that 0·32 of the whole heat would be generated within the central eighth of the volume of the earth, and only one-tenth within 500 miles of the surface. The heat generated at the centre is $3\frac{7}{10}$ times the average, that at the pole $1/2\frac{1}{9}$ of the average, and at the equator $1/12\frac{2}{3}$ of the average; and it turned out that the heat, being so centrically produced, would, on account of the slowness of conduction, not have had time to reach the surface in the 57 million years postulated. This conclusion depending on conduction would of course be true only in the case of a solid earth, the interior of which had the particular viscosity which has been assumed, on which the 57 million years depend.

In connection with this point a serious difficulty seems to arise. Lord Kelvin, in his well-known paper "On the secular cooling of the Earth §," held that, when according to his view it solidified in a comparatively short period of time, the interior was at the temperature of solidification suited to the pressure at every depth, and, because the cooling would not even yet have penetrated to

* "Problems connected with the tides of a viscous spheroid," p. 592.
† "On the precession of a viscous spheroid," p. 494.
‡ p. 561.
§ *Trans. Roy. Soc. Edin.* vol. XXIII. pt. I., p. 157, and *Nat. Phil.*, App. D.

any great depth, it ought to be so still if it is solid. How then, it may be asked, could this enormous amount of heat be perpetually being communicated to the central parts, and they still remain solid? It seems that they must have become heated far above the temperature of fusion appropriate to the pressure, and must now be liquid; as nearly all geologists believe.

I think I have proved in the *Physics of the Earth's Crust** that, if the crust is as thin as geologists suppose, and if the age of the world is anything approaching to what geological phenomena appear to indicate, then there must exist convection currents in the interior, which prevent the crust from growing thicker by melting off the bottom of it nearly as fast as it solidifies. But I made no suggestion to account for such currents being maintained. Here however we appear to find the explanation. This centrically generated heat would be amply sufficient to support fusion, and to keep the currents in action. Indeed the difficulty is rather to see what would become of it all. Darwin's result, regarding the localization of the heat generated, does not depend upon the viscosity, for the coefficient (v) which is introduced into the calculation does not appear in the final result†; but it applies only to the heat generated within the earth by the action of the tidal couple upon the substance of the interior. The distribution of heat within the earth caused by the tidal couple will still follow the same law if only a portion of it is generated within the earth, and the rest within the water of the ocean. Suppose for instance that the earth was either perfectly rigid or perfectly fluid. In either such case no heat would be generated within the earth. But without doubt the friction of the oceanic tidal flow would, in a sufficiently long time, reduce the speed of the rotation‡. The heat in that case would be generated only in the water, and be radiated into space. But besides friction there seems reason to believe that some amount of heat may be generated in the ocean owing to the fact that the speed of the forced tide wave differs from that of the free wave with which a disturbance would travel round the earth under the influence of gravity alone. The question is an interesting one, and the following attempt is made to solve it.

We have ϕ the west longitude of P the place of observation, ωt the moon's angular distance west of the prime meridian. Then the moon is $\phi - \omega t$ east of P.

* 2nd Ed. pp. 77 and 349.

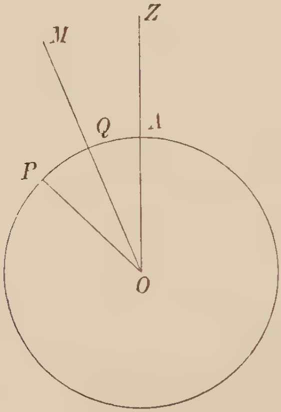
† "Problems connected with tides of a viscous spheroid," p. 558, equation (28), viz.

$$\frac{dE}{dt} = \frac{H}{19}\left[\left\{8 - 5\left(\frac{r}{a}\right)^2\right\} - \frac{3}{2}\left(\frac{r}{a}\right)^2 \sin^2\theta \left\{32 - (26 + \sin^2\theta)\left(\frac{r}{a}\right)^2\right\}\right],$$

where H is the average loss of heat throughout the earth.

‡ Sir W. Thomson on "Geological Time." *Trans. Geol. Soc.* of Glasgow, vol. III. pt. I., 1868, p. 6.

Suppose, as is usual in the canal theory, that AP is developed into a straight line, and that the earth is at rest, and the moon



moving westward above AP. Then, if $AP = x$, the attraction of the moon on the water at P will be in the direction to diminish x, and will be negative. The moon's differential horizontal attraction at P will therefore be

$$-\frac{3aM}{2D^3}\sin 2(\phi - \omega t),$$

which for shortness write $-\mu \sin 2\psi$.

Let, as before, the depth of the canal be h, and its width unity, i.e. one foot, and let y be the height of the tide above the undisturbed water.

Then we have for the accelerations on a unit particle of water at P

$$X = -\mu \sin 2\psi,$$
$$Z = -g.$$

And $$x = a\psi, \quad \therefore\ dx = a\,d\psi.$$

Now the work on a unit particle

$$= \rho \int (X\,dx + Z\,dz)$$

$$= \rho \int (-\mu a \sin 2\psi\, d\psi - g\,dz)$$

$$= \rho \left(\frac{\mu a}{2}\cos 2\psi - gz\right) + C.$$

At an angular distance of 45° from the moon the water is at rest, and its depth is the mean depth of the canal, viz. h. This makes z the depth at P;

$$\therefore \quad 0 = -\rho g h + C.$$

Hence the work on a unit mass of water in the column at P is

$$\rho\left(\frac{\mu a}{2}\cos 2\psi - g(z-h)\right)$$

$$= \rho\left(\frac{\mu a}{2}\cos 2\psi - gy\right)$$

$$= \rho\left\{\frac{\mu a}{2}\cos 2\psi + g\,\frac{a\mu}{2\left(\frac{a^2\omega^2}{h} - g\right)}\cos 2\psi\right\}$$

$$= \rho\,\frac{\mu a}{2}\,\frac{a^2\omega^2}{a^2\omega^2 - gh}\cos 2\psi.$$

Therefore the work on the whole column of unit width is

$$\rho\,\frac{\mu a}{2}\,\frac{a^2\omega^2}{a^2\omega^2 - gh}\cos 2\psi\,(h+y)$$

$$= \rho\,\frac{\mu a}{2}\,\frac{a^2\omega^2}{a^2\omega^2 - gh}\cos 2\psi\left(h - \frac{\mu a}{2}\,\frac{h}{a^2\omega^2 - gh}\cos 2\psi\right)$$

$$= \rho\left\{\frac{\mu a}{2}\,h\,\frac{a^2\omega^2}{a^2\omega^2 - gh}\cos 2\psi - \left(\frac{\mu a}{2}\,\frac{a\omega}{a^2\omega^2 - gh}\right)^2\frac{h}{2}(1+\cos 4\psi)\right\}.$$

To obtain the work done on a length AP of the canal we must multiply this by $a\,d\psi$ and integrate, whence, putting for ψ its value $\phi - \omega t$, and taking the integral from $t = 0$ to $t = t$, we get

$$\text{work on } AP = \rho\left[\frac{\mu a^2}{4}\,h\,\frac{a^2\omega^2}{a^2\omega^2 - gh}\{\sin 2(\phi - \omega t) - \sin 2\phi\}\right.$$

$$+ \left(\frac{\mu a}{2}\,\frac{a\omega}{a^2\omega^2 - gh}\right)^2\frac{ha}{2}\,\omega t$$

$$\left. - \frac{1}{4}\left(\frac{\mu a}{2}\,\frac{a\omega}{a^2\omega^2 - gh}\right)^2\frac{ha}{2}\{\sin 4(\phi - \omega t) - \sin 4\phi\}\right].$$

The work on the whole canal will be given by putting $\phi = 2\pi$, and will be,

$$\text{work on the whole canal} = \rho\left\{-\frac{\mu a^2}{4}\,h\,\frac{a^2\omega^2}{a^2\omega^2 - gh}\sin 2\omega t\right.$$

$$+ \left(\frac{\mu a}{2}\,\frac{a\omega}{a^2\omega^2 - gh}\right)^2\frac{ha}{2}\,\omega t$$

$$\left. + \frac{1}{4}\left(\frac{\mu a}{2}\,\frac{a\omega}{a^2\omega^2 - gh}\right)^2\frac{ha}{2}\sin 4\omega t\right\}.$$

This work is done by the moon upon the whole mass of water, while she traverses the interval AQ.

Hence the work done while she makes a complete revolution will be given by putting $\omega t = 2\pi$, and it will be

$$\rho \left(\frac{\mu a}{2} \frac{a\omega}{a^2\omega^2 - gh}\right)^2 \frac{ha}{2} 2\pi.$$

This work will accumulate once every lunar day.

To obtain the corresponding rise of temperature, we know that a weight m raised through s feet is equivalent to heat sufficient to warm m pounds of water through $s/772$ degrees Fah.; so that to find the rise of temperature produced by the work W upon a mass m of water we have

$$W = mgs,$$

and the equivalent rise of temperature in the water will be

$$\frac{W mg}{772}.$$

In the present instance

$$m = \rho 2\pi ha,$$

and therefore the rise of temperature in the water in a lunar day will be

$$\left(\frac{\mu a}{2} \frac{a\omega}{a^2\omega^2 - gh}\right)^2 \frac{1}{2g772}.$$

We know that

$$\frac{Ma^3}{ED^3} = \frac{1}{18{\cdot}2 \times 10^6}\text{*}.$$

Hence $$\mu = \tfrac{3}{2} \frac{g}{18{\cdot}2 \times 10^6}.$$

And
$$\omega = 0{\cdot}000072924 \text{ radian},$$
$$a = 3959 \text{ miles},$$
$$h = 4 \text{ miles},$$
$$g = 32 \text{ feet per second}.$$

Reducing to feet, the rise of temperature in the water of the equatorial canal in degrees Fah. comes out about

$$0{\cdot}000006^\circ \text{ Fah.}$$

in one year, or 6° in a million years.

We see then that under the present circumstances a very small portion of the heat generated about the earth in this manner would be taken up by the ocean, and radiated into space, irrespective of the friction of the water. But Darwin informs us

* Thomson and Tait, 2nd Ed. p. 383.

that, looking backwards, the moon's orbital velocity increases very rapidly. Now ω is the earth's angular velocity *minus* the moon's orbital velocity. If then retrospectively the moon's orbital velocity increases more rapidly than the earth's angular velocity, ω will diminish.

If we put
$$u = \left(\frac{\mu a}{2} \frac{a\omega}{a^2\omega^2 - gh}\right)^2,$$
then
$$\frac{du}{d\omega} = -\left(\frac{\mu a}{2}\right)^2 \frac{2a^2\,\omega\,(a^2\omega^2 + gh)}{(a^2\omega^2 - gh)^3}.$$

Hence, so long as $a^2\omega^2$ is greater than gh, u will increase as ω diminishes. Moreover the moon's distance diminishes. Hence, $(\mu a/2)^2$ varying inversely as the sixth power of the distance will increase very rapidly; so that on both these accounts the heat generated in the water *per* lunar day will rapidly increase. It must not however be forgotten that the length of the lunar day increases, so that fewer of them go to a year.

The above expression would become infinite if
$$\omega = \sqrt{gh}/a;$$
that is if
$$\omega = 0{\cdot}000039,$$
whereas at the present
$$\omega = 0{\cdot}000073.$$

But such a result cannot be relied upon, because the same value of ω would make the expression for the height of the tide infinite, whereas in the formation of the differential equation from which it is found it has been assumed to be small. But it is evident that the generation of heat in the water must increase as that value of ω is approached, and that something of the nature of a catastrophe will have happened at that juncture, because, going back in time, when that epoch has been passed the expression for the height of the tide is found to have changed signs, and consequently high and low water will have interchanged places then.

If ω is less than $\sqrt{gh}/a$, then $du/d\omega$ becomes positive, and the heat generated in the water rapidly diminishes as ω diminishes.

We know that $\sqrt{gh}$ is the velocity of the free wave, with which a disturbance in the water would be propagated under the influence of gravity alone.

The friction of the tides against the coast-lines will of course have had some effect in retarding the rotation, but how much we cannot estimate*.

We have seen that the fact that the speed of the forced tide wave in the ocean differs from that of the free wave is a cause of

* Airy's "Tides and Waves," § 544. *Encycl. Met.* quoted by Sir W. Thomson, "Geol. Time." *Trans. Geol. Soc.*, Glasgow, 1868.

generation of heat (though small) in the water. A like cause must be in operation within the earth, because the forced bodily tide has a different period from the free gravitational oscillation. The distribution of the additional heat from this cause would probably, if calculated, turn out to be different from that arising from the internal friction produced by the tidal couple, which is the source of internal heat contemplated in Prof. Darwin's work.

It seems then that, unless the ocean tides have been in operation for a length of time exceeding any estimate hitherto suggested, it does not appear probable that any considerable portion of the heat, which according to Darwin's hypothesis of the moon being shed from the earth has been from first to last generated about the earth, can be got rid of by that means. It follows that a largely preponderating amount of it must have accumulated within the earth. This as already remarked must have kept the deeper parts constantly above the temperature of solidification for the pressure, and is an argument in favour of present liquidity.

But if such is their condition we cannot appeal to the slowness of conduction in a solid earth to account for this great amount of heat not making itself evident at the surface; which it must have done unless it has been prevented from accumulating faster than it has been generated. There seem to be only three important means of effecting this, viz. (1) conduction through the solidified crust, (2) transference of heat to the surface by volcanic action, (3) the conversion of heat into work against gravity, and against the molecular forces, expended in modifying during geological ages the condition of the crust.

The first and most obvious mode of escape of heat from the interior, which we now regard as liquid, is by conduction through the solidified crust. I have explained in my *Physics of the Earth's Crust**, how it is the latent heat of the layer by which the crust would have been thickened more than, owing to the action of the hot liquid, it is actually thickened, which escapes by conduction through the crust, and that this heat is abstracted from the interior mass and lowers its temperature. I have also shown that the mean fall of temperature of the interior from this cause, considering the store of heat to have been initial† and the time elapsed

* p. 73.

† It appears however that this assumption is not necessary, for the whole mass remelted (using the symbols in *Physics of the Earth's Crust*) is $y \times 4\pi (r-k)^2$, and it yields up λ times that amount of heat.

This divided by the volume of the interior will give the mean fall of temperature

$$=\frac{\lambda y 4\pi (r-k)^2}{\frac{4}{3}\pi (r-k)^3} = 3\lambda y \frac{1}{r-k},$$

or putting γ for $\frac{y}{k}$ $\qquad = 3\lambda\gamma \frac{k}{r}$ nearly,

as on the hypothesis of the heat being initial.

100 million years, may be put at about 209° Fah. This calculation involves the assumption that the ratio of the rate of thickening to the rate of retardation (or remelting) is constant, or, what is equivalent to this, that the thickness of the crust varies as the square root of the time since it began to be formed. It was there shown* that the assumption of constancy of the above ratio of the rates of thickening and retardation of thickening is probable, because, if their ratio varied as any power of the time, it would lead to unnatural consequences. I now find that the assumption that the store of heat was initial is not necessary, because the same formula can be obtained without that assumption, so long as we adhere to the other assumption that the thickness of the crust varies as the square root of the time, or to its equivalent. Such a calculation is sufficient to show that the amount of heat, carried off by conduction through the crust in an interval even so long as 100 million years, must have been quite inconsiderable compared to the whole amount generated in the interior.

As the most extreme case possible of volcanic action we can estimate approximately the fall of temperature of the earth supposing the whole of the water of the ocean to have been originally in solution with the magma of the interior, and to have carried off a corresponding amount of heat. Professors Rücker and Roberts-Austen have determined the temperature of melting basalt to be about 920° C. † Now the total heat of steam at t degrees C. given off in condensing into water at 0° C. is given by the formula

$$605{\cdot}5 + 0{\cdot}305t\ddagger;$$

whence it appears that unit of vapour at 920° C. will have parted with 886 units of heat in condensing to water at 0° C. Hence every unit mass of the ocean on the hypothesis now made represents 886 units of heat removed from the interior of the globe; for remembering that the main body of water in the great oceans is at very low temperatures, and that a large volume of water at the poles is frozen, it is not a violent supposition to assume 0° C. as the mean temperature.

Now even supposing the ocean to cover the globe and to be four miles deep, its volume will be about 0·003 of the whole globe. The density of the globe is 5·5 that of water. Hence the mass of the ocean is 0·003/5·5 times the mass of the globe. Then, taking as Darwin has done the specific heat of the globe to be that of iron, viz. 1/9, we get the mean temperature of the interior reduced by this means by 4°·53 C. or 8° F., an inappreciable amount compared with the 3000° F. attributed to tidal action, by which the earth is estimated to have been heated.

* Appendix to *Physics of the Earth's Crust*, p. 21.
† "Nature," vol. XLIV., p. 456. Also *Phil. Mag.*, Oct. 1891.
‡ Tait's *Heat*, § 166.

On the above extreme hypothesis that the ocean consists of condensed steam emitted from the interior, the solid ejectamenta of volcanic action would have had a very subsidiary effect in reducing the internal temperature.

There remains the consideration of the heat converted into the work which has been expended in producing elevations of the surface, in shearing and contorting the materials of the crust, and in inducing molecular changes. The amount of this work has no doubt been from first to last enormous; but it is easy to see that a very inconsiderable fall of temperature throughout the interior would represent a very great deal of such work effected. For instance, the work of raising through half its height a layer of granite ten miles thick, weighing 178 pounds per cubic foot, would represent the heat equivalent to a fall of temperature of only one degree F. throughout the globe.

We have not then so far arrived at an answer to the enquiry—What has become of all the heat generated by tidal friction? There appear to be only two replies to this question. One is, that the solidification of the crust took place a very long while subsequent to the genesis of the moon, so that the still liquid surface was able for ages to radiate directly into space the heat carried up to it from below by convection during the time, when, owing to the proximity of the moon, the generation of internal heat went on most rapidly. The other answer can only be, that the moon was not originally thrown off from the earth, but was left behind according to the nebular hypothesis. In that case the whole amount of tidal action would not have been so great, though nevertheless sufficient heat may have been centrically generated by it to maintain those internal currents, which the theory of a thin crust and liquid interior appear to necessitate.

(2) *On Gynodiœcism in the* Labiatae. (First paper.) By J. C. WILLIS, B.A., "Frank Smart" Student in Botany, Caius College.

In July, 1890, my attention was called, by Mr F. Darwin, to the occurrence, on hermaphrodite plants of *Origanum vulgare* in his garden, of occasional flowers having one, two, three, or even *all*, of the stamens aborted. I found such flowers, on examination of many plants, to be of fairly common occurrence. The corolla is usually smaller than in the normal hermaphrodite flower, and may even, especially in the case of the female flowers, be as small as the corolla of a normal female flower (i.e. a flower on a female plant). The aborted stamens are represented by small dark-coloured bodies in the throat of the corolla, usually sessile, but in some cases shortly stalked.

Corresponding variations on the normally female plants were of much rarer occurrence. Occasionally, however, I found a female plant bearing a large hermaphrodite flower among the females, and flowers with one or two stamens also occurred.

That these variations are not simply due to cultivation appears from the fact that they are as common upon the wild form, which I have examined at Abington (Cambs.) and Llangollen. Three batches of plants gathered at the same time in August, 1890, gave the following results:

Batch	Plants	Flowers	I.	II.	III.	IV.	Total	%
A.	12	1146	8	8	14	37	67	5·85
B.	7	745	8	21	11	14	54	7·23
C.	9	588	2	17	6	7	32	5·44
Total	28	2479	18	46	31	58	153	6·17

A. Plants from the "Labiatae" bed, Bot. Gardens, Camb.
B. .... "Medicinal" ....
C. .... Abington (Cambs.)

The numbers in columns I., II., III., IV., represent the numbers of flowers with 1, 2, 3, 4 aborted stamens, respectively.

It will be noticed that the number of flowers fully female is greater than the number of any of the intermediate forms, and this I found to be always the case, if a considerable number of plants were examined. Some of the variations were very striking, e.g. on each of two plants in batch A there occurred a lateral twig which bore female flowers *only*. I have observed the same phenomenon on two or three other occasions.

Similar variations occur upon the hermaphrodite plants of other Labiatae. Müller*, Schulz†, and others have observed them in some, and I have myself noticed them in *Thymus serpyllum*, *Nepeta Glechoma*, and *N. Cataria* (all in the wild state), besides many garden plants of the order, e.g. *Micromeria juliana*, *Nepeta longiflora*, *Hyptis pectinata*, *Bystropogon punctatus*, *Mentha crispa*, *Satureia hortensis* and *S. montana*. The last-mentioned is extremely variable, at least in Cambridge, more than half the flowers usually departing from the normal type.

During 1891 various observations were made upon these abnormalities, with a view to discovering the conditions governing them, and also to throwing some light upon the origin of

* "Fertilisation of Flowers," Eng. Ed. p. 476 (*Calamintha Clinopodium*).

† "Die biologischen Eigenschaften von *Thymus Chamaedrys* Fr. u. *T. angustifolius* Pers." Deutsche Botan. Monatsschr. III. 1885, p. 152.

gynodiœcism. If Ludwig's* view be correct, that the primary cause is the protandry of the flower, rendering the stamens of the earlier flowers useless, we might expect to find these variations more frequent at the commencement of the flowering season. This was tested as follows: Ten cuttings were taken from the same parent stock, and grown under similar conditions: every flower was carefully examined. The abnormalities were most erratic in their occurrence, and I was unable to discover any conditions governing this point. No two of the plants gave results corresponding in any way, nor did the average follow any apparent rule. Some bore about the same percentage of abnormal flowers throughout the season, others bore them many at one time, few at another: e.g. No. 1 bore 126 females altogether; 73 of these appeared in three days, and of these 29 were upon one small lateral branch of the inflorescence. It may however be noted that the percentage of abnormalities was much lower than in 1890, being only 2 per cent.; a result possibly due to the plants being cultivated, and having no competition with one another for space.

One plant was protected from insects by a muslin net throughout the flowering season, and did not, though it bore hundreds of flowers, set a single seed capable of germination. It should be noted, that owing to the smallness of the meshes, the plant could not be shaken by the wind.

Observations were also made (1891) upon *Nepeta Glechoma* (wild). Two lots of plants were examined, one (A) growing on a dry sunny bank, the other (B) in deep shade in a wood. The latter commenced to flower 18 days later than the former and were much taller and less hairy.

The numbers of plants in flower and the numbers of open flowers were counted weekly during the flowering season, and it was found, as Ludwig has observed in the case of thyme, that the proportion of female to hermaphrodite plants in flower was greater at the beginning than at the end of the flowering season. For example in lot A, on the first day 6 female plants and one hermaphrodite flowered; the percentage of females being 85·7. A week later it was 33·6 per cent., and near the end of the season was 23·6 per cent. In lot B, the percentages each week were 50·, 16·, 35·8, 28·5, 23·4, 19·2, 28·3.

It was noticed that the female plants generally bore more open flowers at one time than the hermaphrodites. In lot A the number of flowers on each female plant was (on an average for the whole season) 2·40, and on each hermaphrodite 2·16. In lot B the numbers were 3·15 and 2·16. The greater size of the her-

* "Ueber die Blüthenformen von *Plantago lanceolata* L. und die Erscheinung der Gynodiöcie." Zeitschr. f. d. Ges. Naturw. LII. 1879. p. 441.

maphrodite flowers is thus to some extent compensated for by the greater number of the females.

Abnormalities in the flowers of Nepeta, like those observed in Origanum, &c., are comparatively few and far between, but were yet fairly often encountered.

During the course of these observations upon Nepeta an interesting point was noticed. The protandry of the flowers appears to vary according to the season: at the beginning of the flowering season the stigmas begin to separate very soon after the dehiscence of the anthers, while towards the end of the season these processes are separated by a considerable time. I am conducting further observations upon this point. If it should prove general, it would, taken together with the negative results of the above observations on Origanum, have a tendency to disprove Ludwig's view of the origin of gynodiœcism. This point however I hope to discuss in a future paper, when I shall have concluded the further observations on Origanum, &c., which are now being conducted.

(3) *On the Steady Motion and Stability of Dynamical Systems.* By A. B. Basset, M.A., F.R.S., Trinity College.

1. The object of the present paper is to develop a method for determining the steady motion and stability of dynamical systems, by means of the Principle of Energy, and the Theory of the Ignoration of Coordinates. The subject has already been discussed by Routh*, but is treated in the present paper in a slightly different manner.

Let the coordinates of a dynamical system consist of a group θ, and a group of ignored coordinates χ; and let κ be the constant generalized momentum corresponding to $\dot{\chi}$. Then if the velocities $\dot{\chi}$ be eliminated by means of the equations

$$\frac{dT}{d\dot{\chi}} = \kappa,$$

it is well known that the kinetic energy of the system will be of the form

$$T = \mathfrak{T} + \mathfrak{K},$$

where $\mathfrak{T}$ is a homogeneous quadratic function of the velocities $\dot{\theta}$, and $\mathfrak{K}$ is a similar function of the constant momenta κ.

Also if $\overline{\Theta}$ be that portion of the generalized component of momentum corresponding to $\dot{\theta}$, which does not involve $\dot{\theta}$, and

* *Treatise on Stability of Motion.*

which is consequently a linear function of the κ's, the modified Lagrangian function is

$$L = \mathfrak{T} + \Sigma(\overline{\Theta}\dot{\theta}) - \mathfrak{K} - V \text{..................... (1)},$$

where V is the potential energy, measured from a configuration of stable equilibrium*.

The equations of motion of the system are accordingly

$$\frac{d}{dt}\frac{d\mathfrak{T}}{d\dot{\theta}} + \frac{d\overline{\Theta}}{dt} - \frac{d\mathfrak{T}}{d\theta} - \Sigma\left(\frac{d\overline{\Theta}}{d\theta}\dot{\theta}\right) + \frac{d\mathfrak{K}}{d\theta} + \frac{dV}{d\theta} = 0 \text{ ......(2)}.$$

From this equation it appears, that a steady motion may usually be obtained by assigning constant values to the coordinates θ; whence the equations of steady motion are

$$\frac{d\mathfrak{K}}{d\theta} + \frac{dV}{d\theta} = 0 \text{ ....(3)},$$

where the number of equations of the type (3) is equal to the number of coordinates θ.

2. Let there be m coordinates of the type θ, and n ignored coordinates of the type χ; then we have three cases to consider, according as m is equal to, less than, or greater than n.

Case I. $m = n$.

In this case, the number of equations of the type (3) is equal to the number of momenta κ; hence these equations are sufficient to determine these momenta. Accordingly the conditions of steady motion are, that it should be possible, without violating the connections of the system, to assign constant values to the θ's, such that the values of the m momenta κ, furnished by the solution of (3), should be real.

Case II. $m < n$.

In this case, the number of equations of the type (3) is less than the number of momenta κ. It will therefore be usually possible, when the values of the θ's are given, for the momenta to possess a series of arbitrary values, which lie between certain limits.

Case III. $m > n$.

In this case, the number of equations of the type (3) is greater than the number of momenta κ; it will therefore be possible to eliminate the momenta from (3) in one or more ways. Hence in order that steady motion may be possible, it will be necessary that

* *Proc. Camb. Phil. Soc.*, vol. VI. p. 117; Basset, *Hydrodynamics*, vol. I. p. 174; where, in equation (36), the sign of V ought to be changed.

certain relations should exist between the θ's, or that some of these quantities should have certain definite values.

3. As an example of Case III., let us consider the steady motion of an ellipsoid, which is rotating about its centre of inertia under the action of no forces. In this case, there is one ignored coordinate, viz. ψ; and the momentum corresponding to $\dot{\psi}$ is the constant angular momentum κ about OZ. The value of $\mathfrak{K}$ is

$$\mathfrak{K} = \frac{\frac{1}{2}\kappa^2}{C\cos^2\theta + (A\cos^2\phi + B\sin^2\phi)\sin^2\theta} \quad\text{..............(4).}$$

The equations of steady motion are

$$\frac{d\mathfrak{K}}{d\theta} = 0, \quad \frac{d\mathfrak{K}}{d\phi} = 0 \quad\text{........................(5),}$$

which are satisfied

(i) by $\theta = 0$;
(ii) by $\theta = \frac{1}{2}\pi$, and $\phi = 0$;
(iii) by $\theta = \frac{1}{2}\pi$, and $\phi = \frac{1}{2}\pi$.

These three conditions respectively correspond to rotation about the least, greatest and mean axes. Hence steady motion is impossible, unless the axis of rotation is a principal axis; which is a well-known result.

4. As an example of Case II., we may consider the motion of a solid of revolution or top, spinning about its point. Here ϕ, as well as ψ, is an ignored coordinate; the constant momentum corresponding to $\dot{\phi}$, is the angular momentum $C\omega_3$ of the top about its polar axis. If κ_1, κ_2 be the momenta corresponding to $\dot{\psi}$, $\dot{\phi}$; the value of $\mathfrak{K}$ will be found to be

$$\mathfrak{K} = \frac{(\kappa_1 - \kappa_2\cos\theta)^2}{2A\sin^2\theta} + \frac{\kappa_2^{\ 2}}{2C} \quad\text{........................(6),}$$

also $$V = Mga\,(1 + \cos\theta) \quad\text{........................(7);}$$

whence the equation of steady motion is

$$\frac{d}{d\theta}(\mathfrak{K} + V) = 0 \quad\text{..........................(8),}$$

and will be found to lead to the usual result.

5. We must consider the condition of stability.

Let E be the energy in steady motion, and let the suffixes denote the values of the quantities under these circumstances. Then

$$E = \mathfrak{K}_0 + V_0.$$

Let any disturbance be communicated to the system, and let $E + \delta E$ be the energy of the disturbed motion; then

$$E + \delta E = \mathfrak{T} + \mathfrak{K} + V,$$

whence

$$\delta E = \mathfrak{T} + [(\mathfrak{K} + V) - (\mathfrak{K}_0 + V_0)] \quad \text{..............(9)}.$$

Now $\mathfrak{T}$, being the kinetic energy of a possible motion of the system, is essentially positive, and in the beginning of the disturbed motion must be a small quantity; hence if

$$\mathfrak{K} + V > \mathfrak{K}_0 + V_0,$$

$\mathfrak{T}$ must remain a small quantity, and the system cannot deviate much from its position in steady motion, and the motion will be stable; but if

$$\mathfrak{K} + V < \mathfrak{K}_0 + V_0,$$

$\mathfrak{T}$ may become a finite positive quantity, whilst the term in square brackets may become a finite negative quantity, such that their difference remains equal to the small quantity δE, and the motion may be unstable. Hence the motion will be stable, provided $\mathfrak{K} + V$ is a minimum.

It should be noticed, that this criterion of stability not only includes disturbances which produce variations of the coordinates θ, but also disturbances which produce variations of the momenta κ, though of course the latter quantities always remain constant during the disturbed motion, and equal to their values immediately after disturbance.

6. Routh has shown*, that when there is only *one* coordinate of the type θ, the steady motion will be *unstable* unless $\mathfrak{K} + V$ is a minimum in steady motion; but although we have just shown, that when there are *two* or more coordinates of the type θ, the motion will be stable provided $\mathfrak{K} + V$ is a minimum, it does not follow that the motion will be unstable, when this condition is not satisfied. In fact it sometimes happens, that steady motion will be stable when $\mathfrak{K} + V$ is a maximum. To see this, let us return to the case of the ellipsoid rotating about its centre.

When the axes of rotation are the least, greatest and mean axes, the values of $\mathfrak{K}$ in the beginning of the disturbed motion are respectively equal to

$$\frac{\frac{1}{2}\kappa^2}{C - \{(C-A)\cos^2\phi + (C-B)\sin^2\phi\}\theta^2},$$

$$\frac{\frac{1}{2}\kappa^2}{A + (B-A)\phi^2 + (C-A)e^2},$$

$$\frac{\frac{1}{2}\kappa^2}{B - (B-A)f^2 + (C-B)e^2},$$

* *Stability of Motion*, p. 85.

where θ, ϕ, e, f are small angles. In the first case, $\mathfrak{K}$ is a minimum in steady motion. In the second case, in which rotation takes place about the greatest axis, $\mathfrak{K}$ is a maximum; but it can be shown by other methods, that the steady motion is stable. In the last case, in which the rotation takes place about the mean axis, $\mathfrak{K}$ is a maximum for some disturbances, and a minimum for others; and the motion is well known to be unstable.

The conditions of stability, when there are two coordinates of the type θ, are given by Routh*, and are somewhat complicated. The general conditions of stability, when $\mathfrak{K} + V$ is not a minimum in steady motion, do not appear to have been investigated.

7. The advantages of the Theory of the Ignoration of Coordinates is most strikingly illustrated in Hydrodynamical problems; for in this subject, the most convenient form of the kinetic energy is frequently one, which is not entirely composed of velocities, which are the time variations of coordinates. Moreover, the generalized velocities corresponding to such quantities as components of molecular rotation, vorticity and the like, are frequently unknown, or would be troublesome to introduce. We shall presently call attention to two Hydrodynamical problems, in which the power of this method is strongly brought out.

It must however be noticed, that it is not necessary that the quantities κ should be momenta in the ordinary sense of the word; for if $\tau_1, \tau_2 \ldots$ be any other quantities, which are connected with $\kappa_1, \kappa_2 \ldots$ by a series of relations of the form

$$\kappa_1 = f_2(\tau_1, \tau_2, \ldots),$$

equations (2) and (3) would still apply; provided the functions f do not contain any of the coordinates θ. This remark is of some importance, as a convenient transformation often shortens the work.

8. In the case of a cylinder moving parallel to a fixed wall, when there is circulation†, the kinetic energy is

$$T = \tfrac{1}{2}R(\dot{x}^2 + \dot{y}^2) + \kappa^2 \rho a / 4\pi,$$

where κ is the circulation.

The coordinate x is an ignored coordinate, and the constant momentum h, corresponding to $\dot{x}$, is the momentum of the system parallel to the wall, which is equal to

$$h = R\dot{x} + \kappa\rho c;$$

* *Stability of Motion*, p. 88. † *Hydrodynamics*, vol. I. § 213.

whence $$\mathfrak{K} = \frac{(h - \kappa\rho c)^2}{2R} + \frac{\kappa^2\rho\alpha}{4\pi},$$

and $$V = (M - M')\,gy.$$

The equation of steady motion is

$$\frac{d}{dy}(\mathfrak{K} + V) = 0,$$

which leads at once to

$$pu^2 - \kappa\rho u \coth\alpha + \kappa^2\rho/4\pi c + (M - M')\,g = 0,$$

where $$p = -\tfrac{1}{2}dR/dy, \quad u = \dot{x},$$

which is equation (10), § 213.

The stability of the various cases which arise, can be found from the condition that

$$\frac{d^2}{dy^2}(\mathfrak{K} + V)$$

should be positive, but the calculation would be somewhat troublesome. If however the radius of the cylinder were small in comparison with its distance from the wall, approximate results might be obtained in a fairly simple form.

9. In Hydrodynamical problems, which involve molecular rotation, certain quantities occur, which although not properly speaking momenta, are quantities in the nature of momenta; and to this class of quantities *vorticity* belongs. Let a surface S be drawn in a liquid, cutting each vortex line *once* only; let ω be the resultant molecular rotation, ϵ the angle between the direction of ω and the normal to dS drawn outwards. Then the integral

$$\iint \omega \cos\epsilon\, dS$$

is constant throughout the motion, and this statement expresses the fact that the vorticity of the mass of liquid is constant. The dimensions of this quantity, when multiplied by ρ are $[ML^2T^{-1}]$, and these are the dimensions of an angular momentum. We may therefore regard the constant quantity

$$\rho\iint \omega \cos\epsilon\, dS,$$

as a generalized component of momentum.

When a liquid ellipsoid is rotating about a principal axis (say c), the value of this integral is

$$\pi\rho ab\zeta = \pi\rho R^3\zeta/c,$$

where R is the radius of a sphere of equal volume; whence ζ/c must be constant throughout the motion, and we may therefore treat this quantity, as one which is in the nature of a generalized component of momentum.

The angular momentum h about the axis is also constant; and if we introduce two new constants τ, τ', such that

$$\zeta/c = (\tau' - \tau)/2abc, \quad h = \tfrac{1}{5}M(\tau' + \tau);$$

it follows, that since the volume is constant, we may regard τ', τ as quantities in the nature of generalized components of momentum, and employ them in the place of ζ/c and h. Whence the expression for $\mathfrak{K}$ is

$$\mathfrak{K} = \tfrac{1}{5}M\left\{\frac{\tau^2}{(a-b)^2} + \frac{\tau'^2}{(a+b)^2}\right\},$$

and

$$V = \tfrac{4}{5}M\pi\rho R^2 - \tfrac{2}{5}M\pi\rho abc\int_0^\infty \frac{d\lambda}{(a^2+\lambda)^{\frac{1}{2}}(b^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)^{\frac{1}{2}}};$$

accordingly the steady motion is determined by the equations

$$\frac{d}{da}(\mathfrak{K} + V) = 0, \quad \frac{d}{db}(\mathfrak{K} + V) = 0;$$

in which c must be regarded as a function of a and b. See *Hydrodynamics*, vol. II. §§ 363—367.

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